Math 246B Lecture Notes

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Contents

1 Harmonic Functions

1.1 Relationship to holomorphic functions

We will denote the complex plane as both $\mathbb{R}^2$ with coordinates x_1, x_2 and as $\mathbb{C}$ with complex coordinate $z = x_1 + ix_2$.

Definition 1.1. Let $\Omega \subseteq \mathbb{C}$ be open. We say that $u \in C^2(\Omega)$ is **harmonic** if $\Delta u = 0$ in Ω. Here, $\Omega = \partial_{x_1}^2 + \partial_{x_2}^2 = 4\partial_z \partial_{\overline{z}}$, where

$$\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2}), \qquad \partial_{\overline{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2}).$$

Proposition 1.1. *Let $\Omega \subseteq \mathbb{R}^2$ be simply connected, and let u be real and harmonic. Then $u = \operatorname{Re}(f)$, where $f \in \operatorname{Hol}(\Omega)$, the set of functions $f : \Omega \to \mathbb{C}$ that are holomorphic.*

Proof. Observe that $2\partial_z u$ is holomorphic. So there exists a $g \in \operatorname{Hol}(\Omega)$ such that $g' = \partial_z g = 2\partial_z u$. Then $\partial_z(g + \overline{g}) = 2\partial_z u$. Then $\partial_z(2\operatorname{Re}(g)) = 2\partial_z(2u)$, so $2\operatorname{Re}(g) = 2u + c$ with $c \in \mathbb{R}$. So $u = \operatorname{Re}(g - c)$. □

Remark 1.1. It follows that $u \in C^\infty(\Omega)$ and even real analytic. That is, for any $a \in \Omega$, we have in a neighborhood of a that

$$u(x) = \sum_{j,k=0}^{\infty} c_{j,k}(x_1 - a_1)^j (x_2 - a_2)^k.$$

This is an absolutely convergent power series.

1.2 The Poisson formula and Poisson kernel

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^2$ be open with u harmonic in Ω. If the disc $\{x : |x-a| \leq R\} \subseteq \Omega$, then we have the Poisson formula:*

$$u(x) = \frac{1}{2\pi R} \int_{|y|=R} P_R(x-a, y) u(a+y)\, ds(y), \qquad |x-a| < R.$$

Here, $ds(y)$ is the arc length element along $|y| = R$, and

$$P_R(x, y) = \frac{R^2 - |x|^2}{|x-y|^2}, \qquad |x| < R, |y| = R.$$

Proof. We may assume $a = 0$. Now u is harmonic in $\{|x| < R_1\}$ for some $R_1 > R$. So $u = \operatorname{Re}(f)$, where f is holomorphic in $|z| < R$. Let $|z| < R$, $|w| = R$, and compute:

$$P_R(z, w) = \operatorname{Re}\left(\frac{w+z}{w-z}\right) = \frac{1}{2}\left(\frac{w+z}{w-z} + \overline{\frac{w+z}{w-z}}\right) = \frac{1}{2}\left(\frac{w+z}{w-z} + \frac{R^2 + w\overline{z}}{R^2 - w\overline{z}}\right).$$

Set

$$\varphi_z(w) = \frac{1}{2}\left(\frac{w+z}{w-z} + \frac{R^2 + w\overline{z}}{R^2 - w\overline{z}}\right).$$

If $0 < |z| < R$, then $\varphi_z(0) = 0$. Consider the function $\psi_z(w)$ sending $w \mapsto \varphi_z(w)f(w)/w$ for $|z| < R$.

1. If $0 < |z| < R$, then the singularity at $w = 0$ is removable and the only other singularity in the disc $|w| \leq R$ occurs when $w = z$. It is a simple pole with the residue equals $f(z)/z(1/2)2z = f(z)$.

2. If $z = 0$, $\psi_z(w) = f(w)/w$ has a simple pole at 0, and the residue equals $f(0)$.

For $|z| < R$ and $w = Re^{i\varphi}$, we get $ds(w) = |dw| = R\frac{dw}{iw}$. So we may write

$$\frac{1}{2\pi i}\int_{|w|=R} P_R(z,w)f(w)\,ds(w) = \frac{1}{2\pi i}\int_{|w|=R} \underbrace{P_R(z,w)\frac{f(w)}{w}}_{\psi_z(w)}\,dw = f(z)$$

by the residue theorem. Taking the real part, we get the result. □

Remark 1.2. We can write the Poisson formula as follows:

$$u(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{|Re^{i\tau} - re^{it}|^2} u(Re^{i\tau}\,d\tau = \frac{1}{2\pi}\int_0^{2\pi} \tilde{P}_{R,r}(t-\tau)u(Re^{i\tau})\,d\tau,$$

where

$$\tilde{P}_{R,r}(t) = \frac{R^2 - r^2}{R^2 - 2Rr\cos(t) + r^2}.$$

This is a convolution with the kernel $P_{R,r}(t)$. This function tends as $1/(R-r)$.

Proposition 1.2. *The Poisson kernel $P_R(x,y)$ has the following properties:*

1. $P_R(x,y) \geq 0$.

2. *$x \mapsto P_R(x,y)$ is harmonic for $|x| < R$, $|y| = R$.*

3. *For $|x| < R$,*

$$\frac{1}{2\pi R}\int_{|y|=R} P_R(x,y)\,ds(y) = 1.$$

4. *For all $\varepsilon > 0$ and $\delta > 0$, there exists $R_1 < R$ such that if $|x-y| \geq \delta$ and $R_1 < |x| < R$, then $P_R(x,y) \leq \varepsilon$.*

Proof. For the second property, observe that we expressed the Poisson kernel as the real part of a holomorphic function. For the third, apply the Poisson formula to the harmonic function 1. □

1.3 The Dirichlet problem in the disc

Using the Poisson kernel, we can solve the Dirichlet problem in the disc.

Theorem 1.2. *Let $f \in C(\{x : |x| = R\}; \mathbb{R})$. Then there exists a unique $u \in C(\{|x| \leq \mathbb{R}\})$ such that $u = f$ on $|x| = R$ and u is harmonic in $|x| < R$. The function u is given by*

$$u(x) = \frac{1}{2\pi R} \int_{|y|=R} P_R(x,y) f(y)\, ds(y), |x| < R.$$

Proof. Uniqueness: If u solves the problem, consider $u_\rho(x) = u(\rho(x)$ for $0 < \rho < 1$. The scaled function u_ρ is harmonic near $\{|x| \leq R\}$, so

$$u_\rho = \frac{1}{2\pi R} \int_{|y|=R} P_R(x,y) u_\rho(y)\, ds(y)$$

for $|x| < R$. Keep x fixed and let $\rho \to 1$. We get that

$$u(x) = \frac{1}{2\pi R} \int_{|y|=R} P_R(x,y) f(y)\, ds(y).$$

For existence, define

$$u(x) = \begin{cases} \frac{1}{2\pi R} \int_{|y|=R} P_R(x,y) f(y)\, ds(y) & |x| < R \\ f & x \in \partial D_R. \end{cases}$$

We will give more detail for this part of the proof next time. □

Remark 1.3. We can replace this continuous function f by many things, such as a measure.

2 Mean Value Property and Maximum Principles of Harmonic Functions

2.1 Solving the Dirichlet problem

Last time, given $f \in C(|x| = R)$, we wanted to find a $u \in C^2(|x| < R) \cap C(|x| \le R)$ such that $\delta = 0$ in $|x| < R$ and $u = f$ on $|x| = R$. We defined

$$u(x) = \frac{1}{2\pi R} \int_{|y|=R} P_R(x, y) f(y)\, ds(y), \qquad |x| < R.$$

Then u is harmonic in the disc $|x| < R$, and we need to show that $u \in C(\||x| \le R)$. Let's finish this proof.

Proof. When $0 < \rho < 1$, we let $u_\rho = u(\rho x)$ and show that $u_\rho \to f$ uniformly on $|x| = R$ as $\rho \to 1$. Given $\varepsilon > 0$, let $\delta > 0$ be such that if $|y| = |\tilde{y}| = R$ and $|y - \tilde{y}| \le \delta$, then $|f(y) - f(\tilde{y})| \le \varepsilon$. Let $\rho_1 < 1$ be such that if $|x| = R$, $|y| = R$, and $|x - y| \ge \delta$, then $\rho_1 < \rho < 1 \implies P_R(\rho x, y) \le \varepsilon$. We get

$$\begin{aligned} u_\rho(x) - f(x) &= \frac{1}{2\pi R} \int_{|y|=R} P_R(\rho x, y)(f(y) - f(x))\, ds(y) \\ &= \frac{1}{2\pi R} \left(\int_{\substack{|y|=R \\ |y-x|\le\delta}} + \int_{\substack{|y|=R \\ |y-x|\ge\delta}} \right) \\ &= I_1 + I_2. \end{aligned}$$

Note that $|I_1| \le \varepsilon$. When $\rho_1 < \rho < 1$ we get

$$|I_2| \le \frac{1}{2\pi R} \int_{\substack{|y|=R \\ |y-x|\ge\delta}} P_R(\rho x, y) |f(y) - f(x)|\, ds(y) \le 2M\varepsilon,$$

where $M = \max_{|y|=R} |f(y)|$. We get that

$$|u_\rho(x) - f(x)| \le (1 + 2M)\varepsilon$$

for $\rho_1 < \rho < 1$ and $|x| = R$. Next, if $|x| < R$,

$$|u_\rho(x) - u(x)| = \left| \frac{1}{2\pi R} \int_{|y|=R} P_R(x, y)(u_\rho(y) - f(y))\, ds(y) \right| \le \max_{|y|=R} |u_\rho - f| \xrightarrow{\rho \to 1} 0.$$

We get that $u_\rho \to u$ uniformly on $|x| \le R$, as $\rho \to 1$. The u_ρ are continuous on $|x| \le R$, so $u \in C(|x| \le R)$. $\square$

2.2 Mean value property

Harmonic functions enjoy the following unique continuation principle:

Proposition 2.1. *If $\Omega \subseteq \mathbb{R}^2$ is a domain, $u \in H(\Omega) = \{$harmonic functions on $\Omega\}$, and $u|_\omega = 0$ for nonempty open $\omega \subseteq \Omega$, then u vanishes identically.*

Proposition 2.2 (Mean value property of harmonic functions)**.** *Let $\Omega \subseteq \mathbb{R}^2$ be open, $u \in H(\Omega)$, and $\{|x-a| \leq R\} \subseteq \Omega$. Then*

$$u(a) = \frac{1}{2\pi R} \int_{|y|=R} u(a+y)\, ds(y).$$

Proof. Take $x = a$ in the Poisson formula. □

2.3 Maximum principles of harmonic functions

Theorem 2.1 (maximum principle)**.** *Let $\varnothing \neq \Omega \subseteq \mathbb{R}^2$ be open and bounded with $u \in H(\Omega) \cap C(\overline{\Omega})$. Then for every $x \in \overline{\Omega}$,*

$$\min_{\partial\Omega} u \leq u(x) \leq \max_{\partial\Omega} u.$$

Proof. It suffices to show the result for the maximum; then replace u by $-u$. Let $M = \max_{\overline{\Omega}} u$, and consider the compact set $E = \{x \in \overline{\Omega} : u(x) = M\}$. We have to show that $E \cap \partial\Omega \neq \varnothing$. If $E \cap \partial\Omega = \varnothing$, take $a \in E$ at the smallest positive distance to $\partial\Omega$; this distance exists because E and $\partial\Omega$ are disjoint compact sets. Take $R > 0$ such that $\{|x-a| \leq R\} \subseteq \Omega$. Then $u < M$ on an open arc contained in $\{|x-a| = R\}$. On the other hand, by the mean value property,

$$M = u(a) = \frac{1}{2\pi R} \int_{|y|=R} u(a+y)\, ds(y) < \frac{1}{2\pi R} \int_{|y|=R} M\, ds(y) = M.$$

This is a contradiction. □

There exists a local version of the maximum principle:

Theorem 2.2. *If $u \in H(\Omega)$, where $\Omega \subseteq \mathbb{R}^2$, and u has a local maximum at $a \in \Omega$, then u is constant in the component of a.*

Theorem 2.3 (Hopf's maximum principle)**.** *Let $D = \{|x| < 1\}$ and let $u \in H(D) \cap C(\overline{D})$. Let $x \in \partial D$ be such that $u(x) = \max_{\overline{D}} u$. Then the normal derivative of u at x*

$$N_x = \lim_{t \to 0^-} \frac{u(x+tx) - u(x)}{t} = \lim_{t \to 1^-} \frac{u(tx) - u(x)}{t-1}$$

exists (in the sense that $N_x \in [0,\infty]$*), and*

$$0 \le u(x) - u(z) \le 2\frac{1+|z|}{1-|z|}N_x$$

for $|z| < 1$.

Proof. For $0 < t < 1$, write

$$u(tx) = \frac{1}{2\pi}\int_{|y|=1} P(tx,y)u(y)\,ds(y).$$

So

$$\begin{aligned} u(tx) - u(x) &= \frac{1}{2\pi}\int_{|y|=1} P(tx,y)(u(y)-u(x))\,ds(y) \\ &= \frac{1}{2\pi}\int_{|y|=1} \frac{1-t^2}{|tx-y|^2}(u(y)-u(x))\,ds(y). \end{aligned}$$

Then the difference quotient is

$$\frac{u(tx)-u(x)}{t-1} = \frac{t+1}{2\pi}\int_{|y|=1}\frac{u(x)-u(y)}{|tx-y|^2}\,ds(y).$$

Let $t \to 1$. The first case is when $\liminf_{t\to 1^-}\frac{u(tx)-u(x)}{t-1} < \infty$. By Fatou's lemma,

$$\frac{t+a}{2\pi}\int \liminf_{t\to 1^-}\frac{u(x)-u(y)}{|tx-y|^2}\,ds < \infty.$$

It follows that $y \mapsto u(x) - u(y)/|x-y|^2 \in L^1(\partial D)$. Try to apply dominated convergence to the above:

$$|x-y| \le |tx-y| + |(1-t)x| = |tx-y| + 1 - t \le 2|tx-y|.$$

We get that

$$\frac{u(x)-u(y)}{|tx-y|^2} \le 4\frac{u(x)-u(y)}{|x-y|} \in L^1(y),$$

and by dominated convergence, we get

$$\frac{u(tx)-u(x)}{t-1} \to \frac{1}{\pi}\int_{|y|=1}\frac{u(x)-u(y)}{|x-y|^2}\,ds(y) < \infty.$$

Case 2 is when $\liminf_{t\to 1^-}\frac{u(tx)-u(x)}{t-1} = \infty$. In this case, $N_x = \infty$. We see also that $N_x > 0$ unless u is constant. ☐

Remark 2.1. It follows that $N_x > 0$ unless u is constant.

3 Local Uniform Convergence, Upper Semicontinuity, and Subharmonic Functions

3.1 Local uniform convergence of harmonic functions

Theorem 3.1. *Let $\Omega \subseteq \mathbb{R}^2$ be open, and let $u \in C(\Omega)$ be such that for all $a \in \Omega$, there exists $R_n \to 0$ such that*

$$u(a) = \frac{1}{2\pi R_n} \int_{|y|=R_n} u(a+y)\, ds(y)$$

for all n. Then $u \in H(\Omega)$.

Corollary 3.1. *Let $u_j \in H(\Omega)$ be a sequence such that $u_k \to u$ locally uniformly in Ω. Then $u \in H(\Omega)$, and for every $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, we have $\partial^\alpha u_k \to \partial^\alpha u$ locally uniformly in Ω. Here, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2}$.*

Proof. By the theorem, u has the mean value property, so $u \in H(\Omega)$. If $\{|x-a| \leq R\} \subseteq \Omega$, write (for $|x-a| \leq R/2$)

$$\begin{aligned}\partial^\alpha u_k(x) - \partial^\alpha u(x) &= \frac{1}{2\pi R} \partial_x^\alpha \int_{|y|=R} P_R(x-a, y)(u_k(a+y) - u(a+y))\, ds(y) \\ &= \frac{1}{2\pi R} \int_{|y|=R} \partial_x^\alpha P_R(x-a, y)(u_k(a+y) - u(a+y))\, ds(y).\end{aligned}$$

Here, $|\partial_x^\alpha P_R(x-a, y)| \leq C_{\alpha,R}$ for any $|y| = R$ and $|x-a| \leq R/2$. Therefore,

$$|\partial^\alpha u_k - \partial^\alpha u| \leq C_{\alpha,R} \max_{|y|=R} |u(a+y) - u_j(a+y)| \to 0.$$

Covering a compact set $K \subseteq \Omega$ by finitely many open discs of this form $|x-a| \leq R/2$ for $R = R(a) > 0$, we get that $\partial^\alpha u_k \to \partial^\alpha u$ uniformly on K. $\square$

3.2 Upper semicontinuous functions

Definition 3.1. Let X be a metric space. A function $u : X \to [-\infty, \infty)$ is called **upper semicontinuous** if for every $a \in \mathbb{R}$, the set $\{x \in X : u(x) < a\}$ is open.

Proposition 3.1. *A function $u : X \to [-\infty, \infty)$ is upper semicontinuous if and only if* $\limsup_{y \to x} u(y) \leq u(x)$ *for all $x \in X$.*

Example 3.1. Let $F \subseteq X$ is closed. Then $\mathbb{1}_F$ is upper semicontinuous.

Proposition 3.2. *If u is upper semicontinuous, and $K \subseteq X$ is compact, then u is bounded above, and $\sup_K u$ is achieved.*

Proposition 3.3. *Let $u : X \to [-\infty, \infty)$ be upper semicontinuous and bounded above. Then there exists a sequence $u_j \in C(X)$ such that $u_1 \geq u_2 \geq \cdots \geq u$ and $u_j \to u$ pointwise.*

Example 3.2. Let $\Omega \subseteq \mathbb{C}$ be open, and let $f \in \mathrm{Hol}(\Omega)$. Then $u = \log|f|$ (with $\log(0) = -\infty$) is upper semicontinuous.

3.3 Subharmonic functions

Definition 3.2. Let $\Omega \subseteq \mathbb{R}^2$ be open. We say that a function $u : \Omega \to [-\infty, \infty)$ is **subharmonic** if

1. u is upper semicontinuous.
2. If $K \subseteq \Omega$ is compact and $h \in C(K) \cap H(K^o)$ is such that $u \leq h$ on ∂K, then $u \leq h$ on K.

Example 3.3. If u is harmonic, then by the mean value property, u is subharmonic.

Proposition 3.4. *Let $(u_\alpha)_{\alpha \in A}$ be a family of subharmonic functions on Ω such that $u = \sup_\alpha u_\alpha < \infty$ pointwise and u is upper semicontinuous. Then u is subharmonic. If (u_j) is a decreasing sequence of subharmonic functions, then $u = \lim u_j$ is subharmonic.*

Proof. The first statement is immediate from the definition. For the second statement, first note that that $u = \lim u_j = \inf u_j$ is upper semicontinuous (if u_α is upper semicontinuous for each α, then $\inf_\alpha u_\alpha$ is, as well).

Now let $K \subseteq \Omega$ be compact, let $h \in C(K) \cap H(K^o)$, and let $u \leq h$ on ∂K. Let $\varepsilon > 0$, and let $x_0 \in \partial K$. Then there exists a j such that $u_j(x_0) < u(x_0) + \varepsilon \leq h(x_0) + \varepsilon$. Then $(u_j - h)(x_0)$, where $u_j - h$ is upper semicontinuous on K. So there is a neighborhood V_{x_0} of x_0 such that $u_j(x) - h(x) < \varepsilon$ for all $x \in V_{x_0} \cap \partial K$. Then, for all $k \geq j$, $u_k(x) - h(x) < \varepsilon$ for all $x \in V_{x_0} \cap \partial K$. Covering the compact set ∂K by finitely many open sets of the form V_{x_0}, we get $u_j \leq h + \varepsilon$ on ∂K for all large j. By the subharmonicity of the u_j, we get that $u_j \leq h + \varepsilon$ on K, so $u \leq h$ on K. $\square$

Remark 3.1. This is the same argument as in the standard proof of Dini's theorem in elementary analysis.

Theorem 3.2. *Let $u : \Omega \to [-\infty, \infty)$ be upper semicontinuous. The following are equivalent:*

1. *u is subharmonic*
2. *(local sub-mean value inequality): For every $a \in \Omega$,*

$$u(a) \leq \frac{1}{2\pi R} \int_{|y|=R} u(a+y)\, ds(y)$$

for all small $R > 0$.

3. For every $a \in \Omega$,

$$u(a) \leq \frac{1}{\pi R^2} \iint_{|y| \leq R} u(a+y)\, dy$$

for all small $R > 0$, where dy is Lebesgue measure in $\mathbb{R}^2$.

We will prove these, along with more equivalences, next time.

4 Properties of Subharmonic Functions

4.1 Local conditions equivalent to subharmonicity

Last time, we introduced the notion of a subharmonic function.

Theorem 4.1. *Let $u : \Omega \to [-\infty, \infty)$ be upper semicontinuous. The following are equivalent:*

1. *u is subharmonic.*
2. *If $\{|x - a| \leq R\} \subseteq \Omega$, then*
$$u(a) \leq \frac{1}{2\pi R} \int_{|y|=R} P_R(x - a, y) u(a + y)\, ds(y).$$
3. *(local sub-mean value inequality): For every $a \in \Omega$,*
$$u(a) \leq \frac{1}{2\pi R} \int_{|y|=R} u(a + y)\, ds(y)$$
for all small $R > 0$.
4. *For every $a \in \Omega$,*
$$u(a) \leq \frac{1}{\pi R^2} \iint_{|y|\leq R} u(a + y)\, dy$$
for all small $R > 0$, where dy is Lebesgue measure in $\mathbb{R}^2$.
5. *If $\{|x - a| \leq R\} \subseteq \Omega$, then*
$$u(a) \leq \frac{1}{\pi R^2} \iint_{|y|\leq R} u(a + y)\, dy$$

Remark 4.1. It follows from properties 3 and 4 that subharmonicity is a local property.

Remark 4.2. The integrals in the theorem are Lebesgue integrals of upper semicontinuous functions. If $u : \Omega \to [-\infty, \infty)$ is upper semicontinuous and $K \subseteq \Omega$ is compact, then
$$\int_K u(x)\, dx = \inf_{\substack{u \leq \varphi \\ \varphi \in C(K)}} \int \varphi\, dx \in [-\infty, \infty).$$

Proof. (1) $\implies$ (2): Let $f \in C(|x - a| = R)$, and let $v \in C(|x - a| \leq R)$ be harmonic in $|x - a| < R$ so that $v = f$ along $|x - a| = R$. If $u \leq f$ on $|x - a| = R$, then $u \leq v$ in $|x - a| \leq R$. So
$$u(x) \leq \frac{1}{2\pi R} \int_{|y|=R} P_R(x - a, y) f(a + y)\, ds(y)$$

for $|x-a|<R$. Pick a sequence $f_k \in C(|x-a|=R)$ such that $f_k \downarrow u$. apply this inequality to every function in the sequence, and let $k \to \infty$ by monotone convergence to get the desired inequality.

(2) $\implies$ (3): Take $x = a$.

(2) $\implies$ (5): If $\{|x-a| \leq R\}$, then

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{it})\, dt$$

with $0 < r \leq R$. Multiply by $2r$ and integrate over $[0, R]$. This gives us the area integral, expressed in polar coordinates.

(5) $\implies$ (4): This is a special case.

(3) $\implies$ (1): Let $K \subseteq \Omega$ be compact and $h \in C(K) \cap H(K^o)$ such that $u \leq h$ on ∂K. We want to show that $u \leq h$ on K. The function $u - h$ is upper semicontinuous on K and satisfies the local sub-mean value inequality in K. We can prove the maximum principle for $u - h$ on K with the same proof as for harmonic functions: If $M = \max_K(u-h)$, then the set $\{x \in K : u(x) - h(x) = M\}$ is closed (as $u - h$ is upper semicontinuous on K). We get that $\max_K u - h = \max_{\partial K} \leq 0$. So $u \leq h$ on K.

(4) $\implies$ (1): The argument is similar to the proof of (3) $\implies$ (1), using the local sub-mean value inequality with respect to small discs rather than circles. □

4.2 Mean value property and maximum principle

In the proof of the theorem, we also proved the following property.

Theorem 4.2 (mean value property for subharmonic functions)**.** *Let $\Omega \subseteq \mathbb{R}^2$ be open and bounded, and let u be upper semicontinuous on $\overline{\Omega}$ and subharmonic in Ω. Then*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

We also have the following version of the maximum principle.

Theorem 4.3 (maximum principle for subharmonic functions)**.** *Let $\Omega \subseteq \mathbb{R}^2$ be open and connected, and let u be subharmonic Ω. If u contains a global maximum on Ω, then it is constant.*

Proof. Let $M = \max_\Omega u$, and notice that the sets $\{u < M\}$, $\{u = M\}$ are open. □

It is important to note that the maximum needs to be global. In this sense, subharmonic functions are much less rigid than their harmonic counterparts.

Example 4.1. Here is an example where u attains a local maximum without being constant in Ω. Take $u(z) = \max(0, \mathrm{Re}(z))$.

4.3 Relationship to holomorphic functions

Proposition 4.1. *Let $\Omega \subseteq \mathbb{C}$ be open, and let $f \in \operatorname{Hol}(\Omega)$. Then $u = \log|f| : \Omega \to [-\infty, \infty)$ is subharmonic in Ω.*

Proof. We saw before that u is upper semicontinuous, and we shall check that for all $a \in \Omega$,

$$u(a) \leq \frac{1}{2\pi R} \int_{|y|=R} u(a+y)\, ds(y)$$

for all small $R > 0$. If $f(a) = 0$, then the inequality holds. If $f \neq 0$, then in a small simply connected neighborhood of a, we can write $u = \operatorname{Re}(\log(f))$. Then u is harmonic near a and the inequality holds with an equality for all $R > 0$. $\square$

Next time, we will prove the following result.

Proposition 4.2. *Let $f \in C(|z| \leq R) \cap \operatorname{Hol}(|z| < R)$. Assume that there exists a Lebesgue measurable $E \subseteq \{|z| = R\}$ of positive measure such that $f|_E = 0$. Then $f \equiv 0$ in $|z| < R$.*

5 More Properties of Subharmonic Functions

5.1 Uniqueness of subharmonic functions

Definition 5.1. Denote $SH(\Omega)$ to be the set of all subharmonic functions in Ω.

Last time, we showed that if $u \in SH(\Omega)$ and if $\{|x-a| \le R\} \subseteq \Omega$, then

$$u(x) \le \frac{1}{2\pi R} \int_{|y|=R} P_R(x-a,y) u(a-y)\, ds(y), \qquad |x-a| < R.$$

Now assume that u is upper semicontinuous in $\{|x-a| \le R\}$ and subharmonic in $\{|x-a| < R\}$. Then

$$u(x) \le \frac{1}{2\pi r} \int_{|y|=r} P_r(x-a,y) u(a+y)\, ds(y), \qquad |x-a| < R.$$

To let $r \to R$, we can assume that $u \le 0$ and apply Fatou's lemma. So

$$\begin{aligned} u(x) &\le \limsup_{r \to R} \frac{1}{2\pi r} \int_{|y|=r} P_r(x-a,y) u(a+y)\, ds(y) \\ &\le \frac{1}{2\pi} \int_0^{2\pi} \limsup_{r \to R} \frac{r^2 - |x-a|^2}{|re^{it} - (x-a)|^2} u(a + re^{it})\, dt \\ &\le \frac{1}{2\pi R} \int_{|y|=R} P_R(x-a,y) u(a+y)\, ds(y). \end{aligned}$$

Proposition 5.1. *Let $f \in C(|z| \le R) \cap \mathrm{Hol}(|z| < R)$. Assume that there exists a Lebesgue measurable $E \subseteq \{|z| = R\}$ of positive measure such that $f|_E = 0$. Then $f \equiv 0$ in $|z| < R$.*

Proof. We may assume that $|f| \le 1$. The function $u = \log|f|$ is upper semicontinuous on $|z| = R$, subharmonic in $|z| < R$, so by our previous discussion,

$$\log|f(z)| \le \frac{1}{2\pi R} \int_{|w|=R} \frac{R^2 - |z|^2}{|z-w|^2} \log|f(w)|\, |dw|, \qquad |z| < R.$$

The integrand equals $-\infty$ on E with $m(E) > 0$, so $f \equiv 0$. □

5.2 Local integrability of subharmonic functions

Theorem 5.1. *Let $\Omega \subseteq \mathbb{R}^2$ be open and connected, and let $u \in SH(\Omega)$ with $u \not\equiv -\infty$. Then $u \in L^1_{\mathrm{loc}}(\Omega)$; that is, if $K \subseteq \Omega$ is compact, then $\int_K u(x)\, dx > -\infty$. Furthermore, if $\{|x-a| \le R\} \subseteq \Omega$, then $\int_{|x-a|=R} u(x)\, ds(x) > -\infty$.*

Remark 5.1. The set $\{x \in \Omega : u(x) = -\infty\}$ is a Lebesgue-null set.

Proof. Let E be the set of points $x \in \Omega$ having a neighborhood ω such that $\overline{\omega} \subseteq \Omega$ and $\int_\omega u(x)\, dx > -\infty$. $E \neq \varnothing$ because there exists some $a \in \Omega$ with $u(a) > -\infty$, and the sub-mean value inequality gives

$$u(a) \leq \frac{1}{\pi R^2} \iint_{|x-a|<R} U(x)\, dx$$

for all small $R > 0$. E is also open.

Let us show that $\Omega \setminus E$ is open. If $\Omega \setminus E$ is not open, then there exists $a \in \Omega \setminus E$ and a sequence $a_n \in E$ such that $a_n \to a$. Arbitrarily close to a_n, there exists b_n such that $u(b_n) > \infty$. Picking b_n so that $|a_n - b_n| \to 0$, we get $b_n \to a$ and $u(b_n) > -\infty$ for all n. Take $R > 0$ such that $\{|x-a| < R \subseteq \Omega\}$. Then if $K_n = \{|x - b_n| \leq R/2\}$, we have $K_n \subseteq \Omega$ for large n. So

$$\frac{1}{\pi(R/2)^2} \iint_{K_n} u(x)\, dx \geq u(b_n) > -\infty.$$

For large n, $a \in K_n^o$. So $a \in E$, which contradicts the choice of a. Because Ω is connected, it follows that $\Omega = E$, and therefore $u \in L^1_{\text{loc}}(\Omega)$.

If $\{|x-a| \leq R\} \subseteq \Omega$, write

$$u(x) \leq \frac{1}{2\pi R} \int_{|y|=R} P_r(x-a, y) u(a+y)\, ds(y), \qquad |x-a| < R.$$

We may assume that $u \leq 0$, and then

$$P_R(x-a, y) = \frac{R^2 - |x-a|^2}{|y-(x-a)|^2} \geq \frac{R^2 - \rho^2}{(R+\rho)^2} = \frac{R-\rho}{R+\rho}, \qquad \rho = |x-a|,$$

so

$$u(x) \leq \frac{1}{2\pi R} \frac{R-\rho}{R+\rho} \int_{|y|=R} u(a+y)\, ds(y).$$

This integral must be finite, for otherwise, $u = \infty$ on $|x-a| < R$. □

5.3 Differential characterization of subharmonic functions

Theorem 5.2. *Let $\Omega \subseteq \mathbb{R}^2$ be open, and let $u \in C^2(\Omega, \mathbb{R})$. Then $u \in SH(\Omega)$ if and only if $\Delta u \geq 0$ in Ω.*

Proof. ($\implies$): Taylor expand u at $a \in \Omega$:

$$u(x) = u(a) + u'(a)(x-a) + \frac{1}{2} u''(a)(x-a)(x-a) + o(|x-a|^2),$$

where $u'(a) = (u'_{x_1}(a), u'_{x_2}(a))$ and $u''(a) = (u''_{x_j x_k}(a))_{1 \leq j,k \leq 2}$. Because u is subharmonic, for all small $R > 0$,

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + Re^{it})\, dt.$$

Substituting in the Taylor expansion, the linear terms drop out, and $(x_j - a_j)(x_k - a_k)$ drop out as well, when $j \neq k$. The remaining terms are the diagonal terms, which are exactly given by the Laplacian. So

$$u(a) \leq u(a) + \frac{R^2}{4}\Delta u(a) + o(R^2).$$

We get

$$\frac{R^2}{4}\Delta u(a) + o(R^2) \implies \Delta u(a) \geq 0.$$

($\impliedby$): Assume first that $\Delta u > 0$ in Ω. By the previous computation,

$$\frac{1}{2\pi}\int_0^{2\pi} u(a + Re^{it})\,dt = u(a) + \frac{R^2}{4}\underbrace{\Delta u(a)}_{>0} + o(R^2) > u(a).$$

for small $R > 0$. Thus, $\Delta u > 0 \implies u \in SH(\Omega)$. In general, consider $u_\varepsilon = u + \varepsilon|x|^2$ for $\varepsilon > 0$. Then $\Delta u_\varepsilon \geq 4\varepsilon > 0$, so $u_\varepsilon \in SH(\Omega)$. Letting $\varepsilon \downarrow 0$, we get $u = \lim u_\varepsilon \in SH(\Omega)$. $\square$

6 Subharmonicity and Convexity

6.1 Jensen's inequality and composition of convex functions with subharmonic functions

Last time, we showed that $u \in C^2(\Omega)$ is subharmonic iff $\Delta u \geq 0$ in Ω.

Remark 6.1. Let $u \in SH(\Omega)$ be such that $u \not\equiv$ on any component (so $u \in L^!_{\text{loc}}$). Approximating u by a decreasing sequence of smooth, subharmonic functions, one may show that $\int u \Delta\varphi \, dx \geq 0$ for all $0 \leq \varphi \in C^2(\Omega)$ such that $\varphi = 0$ outside a compact subset of Ω.

Theorem 6.1. *Let Ω be open, $u \in SH(\Omega)$, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be increasing and convex. Then $\varphi \circ u \in SH(\Omega)$ (we define $\varphi(-\infty) = \lim_{t \to -\infty} \varphi(t)$).*

Example 6.1. If $f \in \text{Hol}(\Omega)$, then $|f|^a \in SL(\Omega)$ for any $a > 0$. Write $u = \log|f|$ and $\varphi(t) = e^{at}$, where $a > 0$.

To proof this theorem, we need the following general inequality for convex functions.

Proposition 6.1 (Jensen's inequality)**.** *Let $I \subseteq \mathbb{R}$ be an open interval, and let $\psi : T \to \mathbb{R}$ be convex. Let (Ω, μ) be a measure space equipped with a probability measure ($\mu(\Omega) = 1$). Let $f \in L^1(\Omega, I)$. Then*

$$\psi\left(\int f \, d\mu\right) \leq \int \psi_0 f \, d\mu.$$

Proof. Let $I = (a, b)$, and let $c = \int f \, d\mu \in (a, b)$. If for $a < t_1 < c < t_2 < b$, $c = \alpha t_1 = (1-\alpha)t_2$, where $\alpha = (t_2 - c)/(t_2 - t_1)$, then $\psi(c) \leq \alpha\psi(t_1) + (1-\alpha)\psi(t_2)$. After some algebra, we get

$$\frac{\psi(c) - \psi(t_1)}{c - t_1} \leq \frac{\psi(t_2) - \psi(c)}{t_2 - c}.$$

So

$$\underbrace{\sup_{t_1 < c} \frac{\psi(c) - \psi(t_1)}{c - t_1}}_{= \psi'_{\text{left}}(c)} \leq \underbrace{\inf_{t_2 > c} \frac{\psi(t_2) - \psi(c)}{t_2 - c}}_{= \psi'_{\text{right}}(c)},$$

where these are the left and right derivatives of φ at c. Then $\psi(t) \geq \psi(c) + \psi_{\text{right}}(c)(t - c)$ for all $t \in I$. That is the tangent line at c lies below the graph of ψ. It follows that

$$\int \psi(f) \, d\mu \geq \psi\left(\int f \, d\mu\right) + \psi'_{\text{right}}(c) \cancelto{0}{\left(\int f - c\right)}. \qquad \square$$

Now let's prove the theorem.

Proof. Let $\{|x-a| \leq R\} \subseteq \Omega$. Then

$$u(a) \leq \frac{1}{2\imath R} \int_{|y|=R} u(a+y))\, ds(y).$$

Applying Jensen's inequality,

$$\varphi(u(a)) \leq \frac{1}{2\pi i} \int_{|y|=R} \varphi(u(a+y))\, ds(y).$$

We also check that $\varphi \circ u$ is upper semicontinuous (since φ is continuous). We get that $\varphi \circ u \in SH(\Omega)$. $\square$

6.2 Maximality bounds in an annulus

Theorem 6.2. *Let u be subharmonic in $0 \leq R_1 < |x| < R_2 \leq \infty$, and let $M(r) = \max_{|x|=r} u(r)$. Then $M(r)$ is a convex function of $\log(r) \in (\log(R_1), \log(R_2))$: if $r_1, r_2 \in (R_1, R_2)$ and $0 \leq \lambda \leq 1$, then*

$$M(r_1^\lambda r_2^{1-\lambda}) \leq \lambda M(r_1) + (1-\lambda) M(r_2).$$

If u is subharmonic in $|x| < R$, then $M(r)$ is an increasing function of r.

Proof. We claim that if I is an open interval in $\mathbb{R}$, $f : I \to \mathbb{R}$ is convex if any only if for any compact interval $J \subseteq I$ and any linear function L,

$$\sup_J (f - L) = \sup_{\partial J} (f - L).$$

This follows from the fact that the graph of f on J lies beneath the chord connecting the endpoints.

Using this characterization of convexity, we have to show that if $a, b \in \mathbb{R}$ are such that $\tilde{M}(r) = M(r) = a \log(r) - b$ is such that $M(r_j) \leq 0$ for $j = 1, 2$, then $\tilde{M}(r) \leq 0$ when $r_1 \leq r \leq r_2$. If we set $v(x) = u(x) - a \log|x| - b$, then $v(x) \in SH(R_1 < |x| < R_2)$ since $a \log|x| - b$ is harmonic. Then $\tilde{M}(r) = \max_{|x|=r} v(x)$. If $v(x) \leq 0$ when $|x| = r_1$ and $|x| = r_2$, then $v(x) \leq 0$ for $r_1 \leq |x| \leq r_2$ by the maximum principle. Therefore, $\tilde{M}(r) \leq 0$ for $r_1 \leq r \leq r_2$. This shows that $M(r)$ is convex as a function of $\log(r)$.

If $u \in SH(|x| < R)$, then $M(r)$ increases by the maximum principle applied to u. $\square$

Corollary 6.1 (Hadamard's three circle theorem)**.** *Let $f \in \mathrm{Hol}(R_1 < |z| < r_2)$, and let $M(r) = \max_{|z|=r} |f(z)|$. Then $\log(M(r))$ is a convex function of $\log(r)$: if $r_1, r_2 \in (R_1, R_2)$ and $0 \leq \lambda \leq 1$, then*

$$M(r_1^\lambda r_2^{1-\lambda}) \leq M(r_1)^\lambda M(r_2)^{1-\lambda}.$$

Proof. Apply the theorem to $u = \log|f|$. $\square$

Remark 6.2. This inequality is much sharper than what we get from the usual maximum principle applied to $|f|$: $M(r_1^\lambda r_2^{1-\lambda}) \leq \max(M(r_1), M(r_2))$.

Next time, we will prove the following result (and more).

Proposition 6.2. *If* $u \in SH(|x| < R)$*, then the average*

$$I(r) := \frac{1}{2\pi r} \int_{|y|=r} u(y)\, ds(y).$$

is a convex function of $\log(r)$ *which is increasing.*

7 Averages of Subharmonic Functions

7.1 Convexity of averages of subharmonic functions

Last time, we proved the following theorem.

Theorem 7.1. *If* $u \in SH(R_1 < |x| < R_2$*, then* $M(r) = \max_{|x|=r} u(x)$ *is a convex function of* $\log(r)$.

This gave us a stronger form of the maximum principle. Here is a similar theorem.

Theorem 7.2. *Let* $u \in SH(R_1 < |x| < R_2)$*, let* $0 \leq R_1 < R_2 \leq \infty$*, and let*

$$I(r) = \frac{1}{2\pi r} \int_{|y|=r} u(y)\, ds(y). \qquad R_1 < r < R_2.$$

Then $I(r)$ *is a convex function of* $\log(r)$*. If* $u \in SH(|X| < R)$*, then* $I(r)$ *is increasing, and* $I(r) \xrightarrow{r \to 0^+} u(0)$.

Proof. Write

$$I(r) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{it})\, dt.$$

Approximating u by a decreasing sequence of continuous functions, we see that $I(r)$ is upper semicontinuous. We claim that $I(r)$ satisfies the maximum principle: If $R_1 < r_1 < r_2 < R_2$, then

$$\max_{[r_1, r_2]} I(r) = \max(I(r_1), I(r_2)).$$

Let $R_1 < r_0 < R_2$, and let $\rho > 0$ be small. Let $|x| = r_0$, and write

$$\begin{aligned} u(x) &\leq \frac{1}{\pi\rho^2} \iint_{|y|\leq\rho} u(x+y)\, dy \\ &= \frac{1}{\pi\rho^2} \iint u(x+y) \mathbb{1}_{B_0(\rho)}(y)\, dy \\ &= \frac{1}{\pi\rho^2} \iint u(y) \mathbb{1}_{B_0(\rho)}(y-x)\, dy. \end{aligned}$$

Integrating over $|x| = r_0$, we get

$$\begin{aligned} I(r) &\leq \frac{1}{2\pi r} \frac{1}{\pi\rho^2} \int_{|x|=r_0} \left[\iint u(y) \mathbb{1}_{B_0(\rho)}(y-x)\, dy \right] ds(x) \\ &= \frac{1}{2\pi r} \frac{1}{\pi\rho^2} \iint u(y) \left[\int_{|x|=r_0} \mathbb{1}_{B_0(\rho)}(y-x)\, ds(x) \right] dy \end{aligned}$$

$$= \frac{1}{2\pi r}\frac{1}{\pi\rho^2}\iint u(y)\psi(y)\,dy,$$

where

$$\psi(y) = \int_{|x|=r_0} \mathbb{1}_{B_0(\rho)}(y-x)\,ds(x).$$

The function ψ gives us the 1-dimensional Lebesgue measure of the part of the circle $\{|z-x| = r_0\}$ contained in the ball $B(y,\rho)$. We have

- $\psi \geq 0$,
- ψ is continuous,
- $\psi(y) = \varphi(|y|)$ for some function φ.
- $\varphi(r) = 0$ for $|r - r_0| \geq \rho$
- $\varphi(r_0) > 0$.

We get

$$I(r) \leq \iint u(y)\varphi(|y|)\,dy = \iint_{\substack{0\leq t\leq 2\pi \\ |r-r_0|\leq\rho}} u(re^{it})\varphi(r)r\,dr\,dt = \int \tilde{\varphi}(r)I(r)\,dr,$$

where $\tilde{\varphi}(r) = 2\pi r\varphi(r)$. So

$$I(r_0) \leq \int \tilde{\varphi}(r)I(r)\,dr.$$

If u is harmonic, then equality holds. In particular, using $u = 1$, we get

$$\int \tilde{\varphi}(r)\,dr = 1.$$

The sub-mean value inequality

$$I(r_0) \leq \int \tilde{\varphi}(r)I(r)\,dt$$

can now be used to prove the maximum principle for $I(r)$ in the usual way. This proves the claim.

To show that $I(r)$ is convex, let $R_1 < r_2 < r_2 < R_2$, and let $\tilde{(r)} = I(r) - a\log(r) - b$ be such that $\tilde{I}(r_j) \leq 0$ for $j = 1, 2$. We want to show that $\tilde{I}(r) \leq 0$ when $r_1 \leq r \leq r_2$. This follows from the maximum principle applied to the subharmonic function $u(x) = a\log|x| - b$.

Now assume that u subharmonic in $|x| < R$. We want to show that $I(r)$ is increasing in r. We have $I(r) = f(\log(r))$, where f is convex on $(-\infty, \log(R))$. We want to show that

f is increasing, so it suffices to show that the right derivative $f'_{\text{right}} \geq 0$. If $f'_{\text{right}}(t_0) < 0$ for some t_0, write

$$f(t) \geq f(t_0) + f'_{\text{right}}(t_0)(t - t_0).$$

Letting $t \to -\infty$, we get that $f(t) \to +\infty$. So $I(r) \to +\infty$ as $r \to 0$. This is impossible, as u is locally bounded above.

Finally, we have for all small $r > 0$,

$$u(0) \leq I(r) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{it})\, dt.$$

Using the upper semicontinuity of u at 0, we get that $I(r) \xrightarrow{r \to 0^+} u(0)$. □

Here is a special case of this theorem, applied to a harmonic function u.

Corollary 7.1. *Let u be harmonic in $R_1 < |x| < R_2$. Then*

$$I(r) = a \log(r) + b.$$

Proof. The theorem gives us that

$$\pm I(r) = \frac{1}{2\pi r} \int_{|x|=r} u(x)\, ds(x)$$

are convex functions of $\log(r)$. So $I(r)$ is an affine function of $\log(r)$. □

7.2 The Phragmén-Lindelöf principle

We would like to extend the maximum principle for subharmonic functions to unbounded domains.

Example 7.1. Let $\Omega = \{\mathrm{Im}(z) = x_2 > 0\}$, and let $i(x) = x_2$. This is harmonic, unbounded, and $u|_{\partial\Omega} = 0$. The idea is that we should be ok if we demand that the function does not grow too rapidly at ∞.

We will prove a general theorem which will allow us to do this. The original motivation of Phragmén and Lindelöf was the case of when Ω is a sector of the complex plane.

8 The Phragmén-Lindelöf Principle

8.1 The Phragmén-Lindelöf Principle for subharmonic functions

To prove the Phragmén-Lindelöf[1] principle, let's introduce some notation.

Definition 8.1. Let $\Omega \subseteq \mathbb{R}$ be open and unbounded. We say that $\varphi : \overline{\Omega} \to \mathbb{R}$ is a **Phragmén-Lindelöf function** for Ω if

1. $\varphi(x) > 0$ for large $|x|$.
2. If u is upper semicontinuous on $\overline{\Omega}$, subharmonic in Ω, $u \leq M$ on $\partial\Omega$, and $u(x) \leq \varphi(x)$ for large $x \in \overline{\Omega}$, then $u \leq M$ on $\overline{\Omega}$.

Remark 8.1. Let φ be a PL function for Ω. Let $f \in \mathrm{Hol}(\Omega) \cap C(\overline{\Omega})$ be such that $|f| \leq M$ on $\partial\Omega$ and $|f(z)| \leq e^{\varphi(z)}$ for large $z \in \overline{\Omega}$. Then $|f| \leq M$ on $\overline{\Omega}$.

Given Ω, how do we construct PL functions for Ω?

Theorem 8.1 (Phragmén-Lindelöf principle)**.** *Let $\Omega \subseteq \mathbb{R}^2$ be open and unbounded. Let $\psi : \overline{\Omega} \to [0, \infty)$ be such that*

1. *ψ is lower semicontinuous on Ω ($-\psi$ is upper semicontinuous),*
2. *ψ is super harmonic in Ω ($-\psi$ is subharmonic),*
3. *$\psi(x) \to +\infty$ as $|x| \to \infty$ for $x \in \overline{\Omega}$.*

Let $\varphi > 0$ be such that $\varphi(x) = o(\psi(x))$ when $|x| \to \infty$ for $x \in \overline{\Omega}$. Then φ is a PL function for Ω.

Here is the original argument by Phragmén and Lindelöf.

Proof. Let u be upper semicontinuous on $\overline{\Omega}$, subharmonic in Ω, $u \leq M$ on $\partial\Omega$, and $u(x) \leq \varphi(x)$ for large $x \in \overline{\Omega}$. We want to show that $u \leq M$ on $\overline{\Omega}$. For $\varepsilon > 0$, set $u_\varepsilon = u - \varepsilon\psi$. Then u_ε is upper semicontinuous on $\overline{\Omega}$, subharmonic in Ω, $u_\varepsilon \leq M$ on $\partial\Omega$, and for large $x \in \overline{\Omega}$,

$$u_\varepsilon(x) \leq \varphi(x) - \varepsilon\psi(x) = -\psi(x)\left(\varepsilon - \frac{\varphi(x)}{\psi(x)}\right) \xrightarrow{|x|\to\infty} -\infty.$$

Let $a \in \Omega$, and let $R > |a|$ be such that $u_\varepsilon(x) \leq M$ for $|x| = R$ and $x \in \overline{\Omega}$. If $\Omega_R = \{x \in \Omega : |x| < R\}$, then $\partial\Omega \subseteq \partial\Omega \cup \{x \in \overline{\Omega} : |x| = R\}$, and $u_\varepsilon \leq M$ on $\partial\Omega_R$. Apply the maximum principle to u_ε and the bounded domain Ω_R to get that $u_\varepsilon \leq M$ on Ω_R. So

$$u_\varepsilon(a) = u(a) - \varepsilon\psi(a) \leq M.$$

Letting $\varepsilon \to 0^+$, we get that $u \leq M$ on Ω. So φ is a PL function for Ω. □

[1] Lindelöf was the teacher of Ahlfors.

8.2 Phragmén-Lindelöf for a sector

This important case of the theorem was the original motivation for Phragmén and Lindelöf.

Theorem 8.2 (PL for a sector)**.** *Let* $\Omega = \{z \in \mathbb{C}\setminus\{0\} : \alpha < \arg(z) < \beta\}$ *for* $0 < \beta - \alpha < 2\pi$. *Then* $\varphi(z) = |z|^k$ *is a PL function for* Ω *if* $0 < k < \pi/(\beta - \alpha)$.

Proof. We may assume after a rotation that $\Omega = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \gamma/2\}$, where $0 < \gamma = \beta - \alpha < 2\pi$. Let $k < k_1 < \pi/\gamma$, and consider $\psi(z) = \mathrm{Re}(z^{k_1}) = \mathrm{Re}(e^{k_1 \log(z)})$, using the principal branch of log. This is $\psi(z) = |z|^{k_1} \cos(k_1 \arg(z))$ for $z \in \overline{\Omega}$ with $z \neq 0$. Then ψ is harmonic in Ω, continuous in $\overline{\Omega}$, and $|\psi(z)| \sim |z|^{k_1}$ since $|k_1 \arg(z)| \leq k_1\gamma/2 < \pi/2$. In particular, $\phi = o(\psi)$ at ∞. Therefore, φ is a PL function for Ω. $\square$

Corollary 8.1 (classical PL principle)**.** *Let* $\Omega = \{z \in \mathbb{C} \setminus \{0\} : \alpha < \arg(z) < \beta\}$, *where* $0 < \beta - \alpha < 2\pi$. *Let* $f \in \mathrm{Hol}(\Omega) \cap C(\overline{\Omega})$, *where* $|f| \leq M$ *on* $\partial\Omega$. *Assume that* $|f(z)| \leq C_1 e^{C_2|z|^k}$ *as* $|z| \to \infty$ *for* $z \in \overline{\Omega}$, *where* $0 < k < \pi/(\beta - \alpha)$. *Then* $|f| \leq M$ *on* $\overline{\Omega}$.

Here is an example from the spring 2015 analysis qualifying exam.

Example 8.1. Let $f \in \mathrm{Hol}(\mathbb{C})$ be such that $|f(z)| \leq e^{|z|}$ and $\sup_{x\in\mathbb{R}}(|f(x)|^2 + |f(ix)|^2) < \infty$. Show that f is constant.

Apply the classical Phragmén-Lindelöf principle 4 times, once to each quadrant. Then f is bounded, so f is constant by Liouville's theorem.

8.3 Phragmén-Lindelöf for general domains

Let $\Omega, \tilde{\Omega} \subseteq \mathbb{C}$ be open and unbounded, and let $G : \Omega \to \tilde{\Omega}$ is an an analytic isomorphism such that G extends to a homeomorphism $\overline{\Omega} \to \overline{\tilde{\Omega}}$. Then $|G(z)|$ is large iff $|z|$ is large.Then if φ is a PL function for $\tilde{\Omega}$, $\varphi \circ G$ is a PL function for Ω. (To check this, use that if $u \in SH(\tilde{\Omega})$, then $u \circ G \in SH(\Omega)$.)

Proposition 8.1. *Let* $\Omega = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0, \alpha < \mathrm{Re}(z) < \beta\}$. *Then* $\varphi(z) = e^{k\,\mathrm{Im}(z)}$ *is a PL function for* Ω *for any* $0 < k < \pi/(\beta - \alpha)$.

We will prove this next time. The idea is that we find a conformal map from the half-strip to a sector with a disc removed. The map is $f(z) = e^{-icz}$ for some $0 < c < 2\pi/(\beta-\alpha)$.

9 Phragmén-Lindelöf for Strips and Cauchy's Integral Formula for Non-Holomorphic Functions

9.1 Phragmén-Lindelöf for a half-strip and a strip

Proposition 9.1 (PL for a half-strip)**.** *Let $\Omega = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0, \alpha < \operatorname{Re}(z) < \beta\}$, with $\alpha < \beta$ finite. Then $\varphi(z) = e^{k \operatorname{Im}(z)}$ is a PL function for Ω for any $0 < k < \pi/(\beta - \alpha)$.*

Proof. Let $F(z) = e^{-icz}$, where $c < 2\pi/(\beta - \alpha)$. $F : \Omega \to \tilde{\Omega}$ is conformal, where $\tilde{\Omega} = -\{w \in \mathbb{C} : |w| > 1, c\alpha < \arg(w) < c\beta\}$. F is a homeomorphism $\overline{\Omega} \to \overline{\tilde{\Omega}}$. In $\tilde{\Omega}$, we have the PL function $\varphi(w) = |w|^{k/c}$, where $k/c < \pi/(c(\beta - \alpha))$. We get $\varphi(z) = \tilde{\varphi}(F(z)) = |F(z)|^{k/c} = e^{k \operatorname{Im}(z)}$ is a PL function for Ω. $\square$

Proposition 9.2 (PL for an entire strip)**.** *Let $\Omega = \{z \in \mathbb{C} : \alpha < \operatorname{Re}(z) < \beta\}$ with $\alpha < \beta$ finite. Then $\varphi(z) = e^{k \operatorname{Im}(z)}$ is a PL function for Ω for any $0 < k < \pi/(\beta - \alpha)$. Then $\varphi(z) = e^{k|\operatorname{Im}(z)|}$ is a PL function for Ω for any $0 < k < \pi/(\beta - \alpha)$.*

Proof. Let $u \in SH(\Omega)$ be upper semicontinuous on $\overline{\Omega}$, $u \leq M$ on $\partial\Omega$, and $u(z) \leq \varphi(z)$ for large $z \in \overline{\Omega}$. We want to show that $u \leq M$ on $\overline{\Omega}$. By the previous result, we get that $u \leq \max(M, L)$ on $\Omega_1 = \Omega \cap \{z : \operatorname{Im}(z) > 0\}$, where $L = \max_{[\alpha,\beta]} u < \infty$. Similarly, using $z \mapsto -z$, we conclude that $u \leq \max(M, L)$ on $\Omega_2 = \Omega \cap \{z : \operatorname{Im}(z) < 0\}$. So u is bounded on Ω.

We claim that any positive constant is a PL-function for Ω. It suffices to construct a harmonic $\psi \geq 0$ such that $\psi(z) \to \infty$ as $|z| \to \infty$. We can take $\psi(z) = \operatorname{Re}(\sqrt{z - \gamma})$, where $\gamma < \alpha$. Then $\psi(z) = |z - \gamma|^{1/2} \cos(\arg(z - \gamma)/2) \sim |z^{1/2}|$ at ∞ in Ω. We conclude that $u \leq M$ on $\overline{\Omega}$. So $\varphi(z) = e^{k|\operatorname{Im}(z)|}$ is a PL function for Ω. $\square$

Corollary 9.1 (Hadamard's three line theorem)**.** *Let $\Omega = \{z \in \mathbb{C} : \alpha < \operatorname{Re}(z) < \beta\}$. Let $u \in SL(\Omega)$, upper semicontinuous on $\overline{\Omega}$, $u \leq A$ on $\partial\Omega$, and $u(z) \leq e^{k|\operatorname{Im}(z)|}$ for large $z \in \Omega$, where $0 < k < \pi/(\beta - \alpha)$. Let $M(x) = \sup_{\operatorname{Re}(z)=x} u(z)$ for $\alpha \leq x \leq \beta$. Then M is convex.*

The proof is similar to ideas we've seen before, so we will just give the idea.

Proof. Here is the idea. Let $a, b \in \mathbb{R}$ be such that $\tilde{M}(x) = M(x) - ax - b \leq 0$ for $x = \alpha, \beta$. Show that $\tilde{M}(x) \leq 0$ for $\alpha \leq x \leq \beta$. If $\tilde{u}(z) = u(z) - a\operatorname{Re}(z) - b$, then $\tilde{u} \in SH(\Omega)$ has the right growth at ∞, and $\tilde{M}(x) = \sup_{\operatorname{Re}(z)=x}(z) \implies \tilde{u} \leq 0$ on $\partial\Omega$. By the PL theorem applied to $\tilde{u}$, $\tilde{u} \leq 0$ in Ω. So $\tilde{M}(x) \leq 0$ on $[\alpha, \beta]$. $\square$

9.2 Cauchy's integral formula for non-holomorphic functions

Theorem 9.1 (Cauchy's integral formula for non-holomorphic functions)**.** *Let $\omega \subseteq \mathbb{C}$ be a bounded open set with piecewise C^1 boundary, and let $u \in C^1(\overline{\Omega})$. Then*

$$u(z) = \frac{1}{2\pi i} \int_{\partial\omega} \frac{u(\zeta)}{\zeta - z} \, d\zeta 0 \frac{1}{\pi} \iint_{\omega} \frac{\partial u}{\partial \overline{\zeta}}(\zeta) \frac{1}{\zeta - z} L(d\zeta),$$

where $L(d\zeta)$ is the Lebesgue measure in ω.

Remark 9.1. The integral over ω makes sense, as $1/\zeta \in L^1_{\text{loc}}(\mathbb{C})$:

$$\iint_{|\zeta|<1} \frac{1}{|\zeta|} L(d\zeta) \overset{\zeta = re^{it}}{=} \iint dr \, dt < \infty.$$

Proof. Let $v \in C^1(\overline{\omega})$. By Green's formula,

$$\int_{\partial\omega} v(\zeta) \, d\zeta \overset{\zeta = \xi + i\eta}{=} \int_{\partial\omega} v(\zeta) \, d\xi + iv(\zeta) d\eta = \iint_{\omega} \left(i \frac{\partial v}{\partial \xi} - \frac{\partial v}{\partial \eta} \right) L(d\zeta) = 2i \iint_{\omega} \frac{\partial v}{\partial \overline{z}} \, L(d\zeta).$$

Apply this to $v(\zeta) = u(\zeta)/(\zeta - z)$ and $\omega_\varepsilon = \{\zeta \in \omega : |\zeta - z| > \varepsilon\}$ for small ε. We get

$$\int_{\partial\omega} \frac{u(\zeta)}{\zeta - z} \, d\zeta - \int_{|\zeta - z| = \varepsilon} \frac{u(\zeta)}{\zeta - z} \, d\zeta = 2i \iint_{\omega_\varepsilon} \frac{1}{\zeta - z} \frac{\partial u}{\partial \overline{\zeta}}(\zeta) \, L(d\zeta).$$

Letting $\varepsilon \to 0^+$, we get

$$\int_{|z - \zeta| = \varepsilon} \frac{u(\zeta)}{\zeta - z} \, d\zeta \to 2\pi i u(z),$$

and

$$\iint_{\omega_\varepsilon} \frac{1}{\zeta - z} \frac{\partial u}{\partial \overline{\zeta}} \, L(d\zeta) \to \iint_{\omega} \frac{1}{\zeta - z} \frac{\partial u}{\partial \overline{\zeta}}(\zeta) L(d\zeta) \in L^1$$

by dominated convergence. $\square$

10 Relationships Between Compactly Supported and Holomoprhic Functions

10.1 Solving the inhomogeneous Cauchy-Riemann equation

Last time, we proved the Cauchy integral formula for non-holomorphic functions.

Definition 10.1. When $\Omega \subseteq \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{C}$ is a function, we define the **support** of f $\operatorname{supp}(f) = \overline{\{x \in \Omega : f(x) \neq 0\}}$ (closure with respect to Ω).

Definition 10.2. When $0 \leq k \in \mathbb{N} \cup \{\infty\}$, let $C_0^k(\Omega) = \{u \in C^k(\Omega) : \operatorname{supp}(u) \subseteq \Omega$ is compact$\}$.

Proposition 10.1. *Let $\psi \in C_0^k(\mathbb{C})$. Then there exists $u \in C^k(\mathbb{C})$ solving the inhomogeneous Cauchy-Riemann equation*

$$\frac{\partial u}{\partial \overline{z}} = \psi.$$

Proof. Apply Cauchy's integral formula.

$$\psi(z) = -\frac{1}{\pi} \iint \frac{\partial \psi}{\partial \overline{\zeta}}(\zeta) \frac{1}{\zeta - z} \, L(d\zeta)$$

Make the substitution $\zeta \mapsto \zeta + z$.

$$\begin{aligned} &= -\frac{1}{\pi} \iint \frac{\partial \psi}{\partial \overline{\zeta}}(\zeta + z) \frac{1}{\zeta} L(d\zeta) \\ &= \frac{\partial \psi}{\partial \overline{\zeta}} \left(-\frac{1}{\pi} \iint \frac{\psi(\zeta + z)}{\zeta} L(d\zeta) \right). \end{aligned}$$

We can differentiate under the integral sign because $1/\zeta \in L^1_{\text{loc}}$, and $\psi \in C_0^1$. So we can take

$$u(z) = -\frac{1}{\pi} \iint \frac{\psi(\zeta)}{\zeta - z} \, L(d\zeta) \overset{\zeta \to \zeta + z}{=} \iint \frac{\psi(\zeta - z)}{\zeta} L(d\zeta) \in C^k(\mathbb{C}). \qquad \square$$

10.2 Bounds on derivatives of holomorphic functons

Proposition 10.2. *Let $\Omega \subseteq \mathbb{C}$ be open, and let $K \subseteq \Omega$ be compact. Then there exists $\psi \in C_0^1(\Omega)$ such that $\psi = 1$ in a neighborhood of K.*

Here, ψ is called a **cutoff function**.

Proof. Let $\delta > 0$ be such that $\operatorname{dist}(x, K) \geq \delta$ for any $z \in \mathbb{C} \setminus \Omega$, and let $\tilde{K} = \{z \in \mathbb{C} : \operatorname{dist}(z, K) < \delta/2\}$. $\tilde{K} \subseteq \Omega$ is compact. Let also $\varphi \in C^1(\mathbb{C})$ with $\varphi \geq 0$, $\varphi(z) = 0$ for $|z| \geq 1$, and $\iint \varphi = 1$. For example, we can take

$$\varphi(z) = \begin{cases} B(1 - |z|^2)^2 & |z| \leq 1 \\ 0 & |z| > 1 \end{cases}$$

for some B chosen so that $\iint \varphi = 1$. Let $\varphi_t(z) = t^{-2}\varphi(z/t)$, where $t > 0$. Then $\operatorname{supp}(\varphi_t) \subseteq \{|z| \le t\}$, and $\iint \varphi_t = 1$ for any t.

Now consider

$$\psi(z) = \mathbb{1}_{\tilde{K}} * \varphi_{\delta/3} = \iint \varphi_{\delta/3}(z-\zeta)\mathbb{1}_{\tilde{K}}(\zeta)\, L(d\zeta).$$

Then $\psi \in C^1(\mathbb{C})$. If $\psi(z) \neq 0$, then there exists $\zeta \in \tilde{K}$ such that $|z-\zeta| \le \delta/3$. We get that

$$\operatorname{dist}(z,K) \le \operatorname{dist}(\zeta,K) + |z_\zeta| \le \frac{\delta}{2} + \frac{\delta}{3} \le \frac{5}{6}\delta < \delta.$$

So $\operatorname{supp}(\psi)$ is a compact subset of Ω. That is, $\psi \in C_0^1(\Omega)$. Moreover, for z with $\operatorname{dist}(z,K) \le \delta/12$, $\operatorname{dist}(z - z\zeta, K) \le \operatorname{dist}(z,K) + |\zeta| < \delta/2$, so

$$\psi(z) - 1 = \iint (\mathbb{1}_{\tilde{K}}(\zeta) - 1)\varphi_{\delta/3}(z-\zeta)\, L(d\zeta) = \iint (\mathbb{1}_{\tilde{K}}(z-\zeta) - 1)\varphi_{\delta/3}(\zeta) L(d\zeta) = 1. \quad \square$$

Remark 10.1. This construction is valid in any Euclidean space, not just $\mathbb{C}$.

Proposition 10.3. *Let $f \in \operatorname{Hol}(\Omega)$. For any compact $K \subseteq \Omega$ and any open neighborhood $\omega \subseteq \Omega$ of K, we have for $j = 0, 1, 2, \dots$ that there exists a constant $C_j = C_{j,\omega,K}$ such that*

$$\sup_{z \in K} |f^{(j)}(z)| \le C_j \|f\|_{L^1(\omega)}.$$

Proof. Let ψ be as in the previous proposition. Apply Cauchy's integral formula to the function $\psi f \in C_0^1(\Omega) \subseteq C_0^1(\mathbb{C})$:

$$(\psi f)(z) = -\frac{1}{\pi} \iint \underbrace{\frac{\partial}{\partial \bar{\zeta}}(\psi f)(\zeta)}_{= \frac{\partial \psi}{\partial \bar{\zeta}} f} \frac{1}{\zeta - z}\, L(\zeta)$$

for all $z \in \mathbb{C}$. So for z in a neighborhood of K,

$$f(z) = -\frac{1}{\pi} \iint \frac{\partial \psi}{\partial \bar{\zeta}}(\zeta) \frac{f(\zeta)}{\zeta - z} L(d\zeta).$$

where the region of integration is $\operatorname{supp}(\frac{\partial \psi}{\partial \bar{\zeta}}) \cap K$. Differentiating under the integral sign, we get

$$f^{(j)}(z) = -\frac{j!}{\pi} \iint \frac{\partial \psi}{\partial \bar{\zeta}}(\zeta) \frac{f(\zeta)}{(\zeta - z)^{j+1}} L(d\zeta).$$

So

$$\|f^{(j)}\|_{L^\infty(K)} \le \frac{j!}{\pi \delta^{j+1}} \left\| \frac{\partial \psi}{\partial \bar{\zeta}} \right\|_{L^\infty} \|f\|_{L^1(\omega)},$$

where $|\zeta - z| \ge \delta$. $\square$

11 Runge's Theorem and Compact Exhaustion

11.1 Runge's theorem

Last time, we showed that if $\Omega \subseteq \mathbb{C}$ is open, $K \subseteq \Omega$ is compact, and $f \in \mathrm{Hol}(\Omega)$, then

$$f(z) = -\frac{1}{\pi} \iint \frac{\partial \psi}{\partial \overline{\zeta}} \frac{f(\zeta)}{\zeta - z} L(ds),$$

where $\psi \in C_0^1(\Omega)$ and $\psi = 1$ near K.

Let $\Omega \subseteq \mathbb{C}$ be open, and let $\tilde{\Omega} \subseteq \Omega$ be a connected component of Ω. Then $\tilde{\Omega}$ is open, and $\partial \tilde{\Omega} \subseteq \partial \Omega \subseteq \mathbb{C} \setminus \Omega$.

Example 11.1. Let $K \subseteq \mathbb{C}$ be compact, and let $\Omega = \mathbb{C} \setminus K$. Then Ω has precisely 1 unbounded component. Indeed, if $R > 0$ is large, then $\{|z| > R\} \subseteq \Omega$ is connected, so it is contained in a single component.

Theorem 11.1 (Runge)**.** *Let $K \subseteq \mathbb{C}$ be compact, and let $A \subseteq \mathbb{C}$ be such that any bounded component of $\mathbb{C} \setminus K$ intersects A. Let f be holomorphic in a neighborhood of K. Then for every $\varepsilon > 0$, there is a rational function $r(z) = p(z)/q(z)$ with p, q polynomials and $q(z) \neq 0$ (when $z \notin A$) such that $|f(z) - r(z)| \leq \varepsilon$ for all $z \in K$.*

Proof. We can use the previous formula for f, where Ω is our neighborhood of K where f is holomorphic. Approximate the right hand side by a Riemann sum of the form

$$g(z) = \sum_j \frac{a_j}{\zeta_j - z},$$

where $\zeta_j \in \mathbb{C} \setminus K$. Then approximate each $1/(\zeta_j - z)$ by a rational function as in the theorem, using a "pole-pushing" argument. By approximating with suitable polynomials, we can "push" the pole from ζ_j to another point outside of A. $\square$

Corollary 11.1 (Runge's theorem for polynomials)**.** *Let $K \subseteq \mathbb{C}$ be compact and simply connected, and let f be holomoprhic in a neighborhood of K. Then f can be approximated by polynomials in z, uniformly on K.*

Remark 11.1. The condition that A meets every bounded component of $\mathbb{C} \setminus K$ is necessary. Let V be a bounded component of $\mathbb{C} \setminus K$, ket $a \in V$, and let $f(z) = \frac{1}{z-a}$ be holomorphic in a neighborhood of K. Assume that for every $\varepsilon > 0$, there exists r(z) rational with no poles in V such that $|f(z) - r(z)| \leq \varepsilon$ on K. Then $|1 - (z-a)r(z)| \leq C\varepsilon$ for all $z \in K$. Now $\partial V \subseteq K$, so, by the maximum principle, $|1 - (z-a)r(z)| \leq C\varepsilon$ for all $z \in V$. This is a contradiction when we set $z = a$.

Definition 11.1. Let $\Omega \subseteq \mathbb{C}$ be open, and let $\omega \subseteq \Omega$ be open. Then ω is **relatively compact** if $\overline{\omega}$ is a compact subset of Ω.

Corollary 11.2. *Let $\Omega \subseteq \mathbb{C}$ be open, and let $K \subseteq \Omega$ be compact. Assume that no component of $\Omega \setminus K$ is relatively compact in Ω. Then any function holomorphic in a neighborhood of K can be approximated uniformly on K by functions in* $\mathrm{Hol}(\Omega)$*.*

Proof. In view of Runge's theorem, we only need to check that if O is a bounded component of $\mathbb{C} \setminus K$, then $O \cap (\mathbb{C} \setminus \Omega) \neq \varnothing$. Indeed, if $O \subseteq \Omega$, then $\overline{O} \subseteq \Omega$. Here, $\overline{O}$ is compact, and O is a component of $\Omega \setminus K$. □

11.2 Compact exhaustion

Proposition 11.1 (compact exhaustion with good properties)**.** *Let $\Omega \subseteq \mathbb{C}$ be open. There exist compact sets $K_n \subseteq \Omega$ such that*

1. *$K_n \subseteq K_{n+1}$ for $n = 1, 2, \dots$.*
2. *$\bigcup_{n=1}^\infty K_n = \Omega$.*
3. *Every bounded component of $\mathbb{C} \setminus K_n$ intersects $\mathbb{C} \setminus \Omega$.*

Proof. Set $K_n = \{z \in \mathbb{C} : |z| \leq n, \mathrm{dist}(z, \mathbb{C} \setminus \Omega) \geq 1/n\}$. Then we have the first two properties. Let us check that each bounded component of $\mathbb{C} \setminus K_n$ contains a bounded component of $\mathbb{C} \setminus \Omega$.

$$\begin{aligned} \mathbb{C} \setminus K_n &= \{|z| > n\} \cup \{z : \mathrm{dist}(z, \mathbb{C} \setminus \Omega) < 1/n\} \\ &= \{|z| > n\} \cup \bigcup_{a \in \mathbb{C} \setminus \Omega} D(a, 1.n). \end{aligned}$$

Let O be a bounded component of $\mathbb{C} \setminus K_n$. Then $O \subseteq \bigcup_{a \in \mathbb{C} \setminus \Omega} D(a, 1/n)$. Thus, there exists $a \in \mathbb{C} \setminus \Omega$ such that $D(a, 1/n) \subseteq \Omega$. Let V be the component of $\mathbb{C} \setminus \Omega$ such that $a \in V$. Then $V \subseteq \mathbb{C} \setminus \Omega \subseteq \mathbb{C} \setminus K_n$ is connected, and $V \cap O \neq \varnothing$. Thus, $V \subseteq O$, so V is bounded. □

Next time, we will show that if $f \in \mathrm{Hol}(\Omega)$, there exist rational r_n, holomoprhic in Ω, such that $r_n \to f$ locally uniformly.

12 Applications of Runge's Theorem

12.1 Locally uniform approximation of holomorphic functions

Last time, we showed that if $\Omega \subseteq \mathbb{C}$, we can find an increasing sequence $K_n \subseteq \Omega$ of compact sets such that $\Omega = \bigcup_{n=1}^\infty K_n$ and such that every bounded component of $\mathbb{C} \setminus K_n$ contains a bounded component contains a bounded component of $\mathbb{C} \setminus \Omega$.

Corollary 12.1. *Let $\Omega \subseteq \mathbb{C}$ be open, and let $A \subseteq \mathbb{C} \setminus \Omega$ be such that each bounded component of $\mathbb{C} \setminus \Omega$ meets A. Let $f \in \mathrm{Hol}(\Omega)$. Then there exist rational functions r_n that have no poles outside of A such that $r_n \to f$ locally uniformly in Ω. If $\mathbb{C} \setminus \Omega$ has no bounded component,, then there exists a sequence of polynomials p_n such that $p_n \to f$ locally uniformly in Ω.*

Proof. Let (K_n) be a compact exhaustion as before. By Runge's theorem and the property of the compact exhaustion, for every n, there exists a rational function r_n with no poles outside of A such that $|f - r_n| \leq 1/n$ on K_n. Since any compact $K \subseteq K_N \subseteq K_n$ for large $n \geq N$, we get $r_n \to f$ uniformly on K.

If $\mathbb{C} \setminus \Omega$ has no bounded component, then none of the sets $\mathbb{C} \setminus K_n$ has a bounded component. By Runge's theorem, for any n, there is a polynomial p_n such that $|f - p_n| \leq 1/n$ on K_n. So $p_n \to f$ locally uniformly in Ω. $\square$

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the one-point compactification of $\mathbb{C}$.

Corollary 12.2. *Let $\Omega \subseteq \mathbb{C}$ be such that $\hat{\mathbb{C}} \setminus \Omega$ is connected. Let $f \in \mathrm{Hol}(\Omega)$. Then there exist polynomials p_n such that $p_n \to f$ locally uniformly.*

Proof. If suffices to show that $\mathbb{C} \setminus K_n$ has no bounded component for all n. For contradiction, let V be a bounded component of $\mathbb{C} \setminus K_n$. Then there is a bounded component C of $\mathbb{C} \setminus \Omega$ such that $C \subseteq V$. In particular, $(\hat{\mathbb{C}} \setminus \Omega) \cap V \neq \varnothing$. Let $V' \subseteq \hat{\mathbb{C}}$ be the union of all the other components of $\mathbb{C} \setminus K_n$ (including the unbounded one) and $\{\infty\}$. Then $V \cap V' = \varnothing$, V and V' are open in $\hat{\mathbb{C}}$, and $V \cup V' \supseteq \hat{\mathbb{C}} \setminus \Omega$: $(\hat{\mathbb{C}} \setminus \Omega) \cap V \neq \varnothing$, and $(\hat{\mathbb{C}} \setminus \Omega) \cap V' \neq \varnothing$ (because ∞ is in the intersection). This contradicts the assumption that $\hat{\mathbb{C}} \setminus K_n$ is connected. $\square$

12.2 Solving the inhomogeneous Cauchy-Riemann equation

Earlier, we solved the inhomogeneous Cauchy-Riemann equation for functions which are compactly supported. We even had a formula for it. Let's show a related result for non-compactly supported funcitons.

Theorem 12.1. *Let $\Omega \subseteq \mathbb{C}$ be open, and let $f \in C^1(\Omega)$. Then there exists $u \in C^1(\Omega)$ such that $\frac{\partial u}{\partial \bar{z}} = f$ in Ω.*

Proof. Let $(K_j)_{j\geq 1}$ be a compact exhaustion of Ω, as before. Let $\psi_j \in C_0^1(\Omega)$ be such that $0 \leq \psi_j \leq 1$ and $\psi_j = 1$ near K_j. Let

$$\varphi_j = \begin{cases} \psi_j - \psi_{j-1} & j > 1 \\ \psi_j & j = 1. \end{cases}$$

Then $\varphi_j \in C_0^1(\Omega)$, $\varphi_j = 0$ in a neighborhood of K_{j-1}, and sum $\sum_{j=1}^\infty \varphi_j$ has only finitely many nonzero terms for each $x \in \Omega$ (and hence converges). We can calculate

$$\sum_{j=1}^\infty \varphi_j = \lim_{N\to\infty} \sum_{j=1}^N \varphi_j = \lim_{N\to\infty} (\psi_1 + \sum_{j=2}^N (\psi_j - \psi_{j-1})) = \lim_{N\to\infty} (\psi_1 + \psi_N - \psi_1) = 1.$$

This is called a **locally finite paritition of unity**. Write $f = \sum_{j=1}^\infty \varphi_j f$, where $\varphi_j f \in C_0^1(\Omega) \subseteq C_0^1(\mathbb{C})$. As $u_j f$ is compactly supported, there exists a function $u_j \in C^1(\mathbb{C})$ such that $\frac{\partial u_j}{\partial \bar{z}} = \varphi_j f$ (we can take $u_j(z) = (1/\pi) \iint \varphi_j f(\zeta)/(z-\zeta)\, L(ds)$).

Here is the problem: the sum $\sum_j u_j$ may not converge. We know that $\frac{\partial u_j}{\partial \bar{z}} = 0$ in a neighborhood of K_{j-1}, so u_j is holomorphic near K_{j-1}. By Runge's theorem, there exists a function $v_j \in \mathrm{Hol}(\Omega)$ such that $|u_j - v_j| \leq 2^{-j}$ on K_{j-1} for all j. Now try the sum $u = \sum_{j=1}^\infty (u_j - v_j)$. We claim that $u \in C^1(\Omega)$ and $\frac{\partial u}{\partial \bar{z}} = f$. Let $K \subseteq \Omega$ be compact, and let N be such that $K \subseteq K_N$. Then

$$u = \sum_{j=1}^N (u_j - v_j) + \sum_{j=N+1}^\infty (u_j - v_j),$$

and $|u_j - v_j| \leq 2^{-j}$ on K, so $u \in C(\Omega)$. Since $\partial_{\bar{z}}(u_j - v_j) = 0$ in a neighborhood of K_{j-1}, $u_j - v_j$ is holomorphic in a neighborhood of K_N, where $j \geq N+1$. So the sum of the series $\sum_{j=N+1}^\infty (u_j - v_j)$ is holomorphic in K_N. Thus, $u \in C^1(\Omega)$, and we compute in K_N^o:

$$\frac{\partial}{\partial \bar{z}} = \sum_{j=1}^N \partial_{z_j}(u_j - v_j) = \sum_{j=1}^N \varphi_j f = \left(\sum_{j=1}^N \varphi_j + \underbrace{\sum_{j=N+1}^\infty \varphi_j}_{=0 \text{ in } K_N} \right) f = f. \qquad \square$$

13 Mittag-Leffler's Theorem and Infinite Products of Holomorphic Functions

13.1 Mittag-Leffler's theorem

Last time, we showed that if $\Omega \subseteq \mathbb{C}$ is open and $f \in C^1(\Omega)$, then there exists some $u \in C^1(\Omega)$ such that $\frac{\partial u}{\partial \overline{z}} = f$ in Ω. Here is an application.

Theorem 13.1 (Mittag-Leffler)**.** *Let $\Omega \subseteq \mathbb{C}$ be open, and let $A \subseteq \Omega$ be a set with no limit points in Ω. For each $a \in A$, let p_a be a rational function of the form*

$$p_a(z) = \sum_{j=1}^{N_a} \frac{c_{a_j}}{(z-a)^j}$$

for some c_{a_j} where $1 \leq N_a < \infty$. Then there exists a $f \in \mathrm{Hol}(\Omega \setminus A)$ such that for all $a \in A$, $f - p_a$ is holomorphic in a neighborhood of a.

Remark 13.1. In other words, f is a meromorphic function in Ω with poles only in A, and for any $a \in A$, p_a is the singular part of the Laurent expansion of f at a.

Proof. The idea is to solve the problem first in the smooth (C^1) category and then correct a smooth solution to get a holomorphic solution solving a $\overline{\partial}$-problem.

The set A is at most countable, and we may assume A is infinite: $A = \{a_1, a_2, \dots\}$. Let $U_j \subseteq \Omega$ be a small neighborhood of a_j such that $\overline{U}_j \cap \overline{U}_\ell = \varnothing$ for $j \neq \ell$, and let $\varphi_j \in C_0^k(U_j)$, where $k \geq 2$, be such that $\varphi_j = 1$ in a neighborhood of a_j. Define

$$g(z) = \sum_{j=1}^{\infty} p_{a-j}(z)\varphi_j(z)$$

for $z \in \Omega \setminus A$. For every compact $K \subseteq \Omega$, $U_j \cap K = \varnothing$ for all but finitely many j. So $g \in C^k(\Omega \setminus A)$, and near a_j, $g - p_{a_j} \equiv 0 \in C^K$.

Next, compute

$$\frac{\partial g}{\partial \overline{z}} = \sum_{j=1}^{\infty} \frac{\partial}{\partial \overline{z}}(p_{a_j}\varphi_j) = \sum_{j=1}^{\infty} p_{a_j} \frac{\partial \varphi_j}{\partial \overline{z}},$$

which is 0 near a_j for any j. Since $\frac{\partial g}{\partial \overline{z}} = 0$ on A, $\frac{\partial g}{\partial \overline{z}}$ extends to a C^{k-1} function on Ω: $\frac{\partial g}{\partial \overline{z}} \in C^{k-1}(\Omega) \subseteq C^1(\Omega)$. Now let $u \in C^1(\Omega)$ be such that $\frac{\partial u}{\partial \overline{z}} = \frac{\partial g}{\partial \overline{z}}$ in Ω. Define $f(z) = g(z) - u(z) \in C^1(\Omega \setminus A)$. Then $\overline{\partial} f = 0$, so $f \in \mathrm{Hol}(\Omega \setminus A)$. In a neighborhood of $a_j \in A$, we write

$$f - p_{a_j} = \underbrace{g - p_{a_j}}_{\in C^k \text{ near } a_j} - \underbrace{u}_{\in C^1}.$$

Then $f - p_{a-j}$ is bounded in a set of the form $0 < |z - a_j| < r_j$ for small r_j, so $f - p_j$ has a removable singularity at a_j. So $f - p_{a_j}$ is holomorphic near a_j for all j. $\square$

13.2 Infinite products of holomorphic functions

Next, we will discuss Weierstrass's theorem, which basically says that any subset of $\Omega \subseteq \mathbb{C}$ with no limit points in Ω is the zero set of some holomorphic function. The idea is to try infinite products of holomorphic functions. You can see how Mittag-Leffler's theorem is inspired by this result.[2]

Proposition 13.1. *Let $\Omega \subseteq \mathbb{C}$ be open, and let (f_j) be a sequence in $\mathrm{Hol}(\Omega)$. Assume that for every compact $K \subseteq \Omega$, there exists $N \in \mathbb{N}$ and a convergence series $\sum_{j=N}^\infty M_j < \infty$ with $M_j \geq 0$ such that f_j is nonvanishing on K for all $j \geq N$ and that $|\mathrm{Log}(f_j(z))| \leq M_j$, wher $j \geq N$, and $z \in K$. This is the principal branch of $\log$: $\arg \in (-\pi, \pi]$. Then the sequence $(\prod_{j=1}^n f_j)$ converges locally uniformly in Ω, $f(z) := \lim_{n \to \infty} \prod_{j=1}^n f_j(z) \in \mathrm{Hol}(\Omega)$, and we write $f(z) = \prod_{j=1}^\infty f_j(z)$. The zeros of f are given by the union of the zeros of the f_j, counting multiplicities.*

Proof. Let $K \subseteq \Omega$ be compact, and let N, M_j be as in the proposition. For $j \geq N$, write $f_j = e^{\mathrm{Log}(f_j)}$. Then

$$\prod_{j=N}^n f_j = \exp \underbrace{\left(\sum_{j=N}^n \mathrm{Log}(f_j) \right)}_{\text{converges uniformly on } K},$$

so, using $|e^z - e^w| \leq e^{\max(\mathrm{Re}(z), \mathrm{Re}(w))}|z-w|$, we write

$$\left| \prod_{j=N}^n - \prod_{j=N}^m f_j \right| \leq C_K \sum_{j=n+1}^m |\mathrm{Log}(f_j)| \to 0$$

uniformly on K. To show that $|e^z - e^w| \leq e^{\max(\mathrm{Re}(z), \mathrm{Re}(w))}|z-w|$, note that

$$e^z - e^w = \int_0^1 \frac{f}{dt} e^{tz+(1-t)w}\, dt. \qquad \square$$

Example 13.1. Assume that $(f_j) \in \mathrm{Hol}(\Omega)$ is such that for every compact $K \subseteq \Omega$, we have $\sum_{j=1}^\infty \sup_K |1-f_j| < \infty$ (normal convergence on each compact). Then the proposition applies, and the product $\prod_{j=1}^\infty f_j$ converges locally uniformly in Ω.

[2] Mittag-Leffler was a student of Weierstrass.

14 Weierstrass's Theorem

14.1 Constructing holomorphic functions with a given zero set

Here is Weierstrass's theorem, which allows us to construct holomorphic functions with a prescribed zero set.

Theorem 14.1 (Weierstrass)**.** *Let $\Omega \subseteq \mathbb{C}$ be open, and let $A \subseteq \Omega$ be a set with no limit point in Ω. Assume that for any $a \in A$, we are given a positive integer $n(a)$. There exists $f \in \mathrm{Hol}(\Omega)$ such that $f^{-1}(\{0\}) = A$, and the multiplicity of each $a \in A$ is $n(a)$.*

Proof. We may assume that A is infinite and write $A = \{a_k, k = 1, 2, \dots\}$ with $a_k \neq a_{k'}$ if $k \neq k'$. Call $n_k := n(a_k)$. We shall try to construct f as an infinite product of the form

$$\prod_{k=1}^{\infty} (z - a_k)^{n_k} e^{g_k(z)},$$

where $g_j \in \mathrm{Hol}(\Omega)$ are chosen to achieve convergence.

Introduce the compact exhaustion $K_j = \{z \in \mathbb{C} : |z| \leq j, \mathrm{dist}(z, \mathbb{C} \setminus \Omega) \geq 1/j\}$. For each k, we have $a_k \in K_j$ for all j large enough. Define the sequence

$$j(k) = \begin{cases} 1 & a_k \in K_1 \\ \max\{j : a_k \notin K_j\} & a_k \notin K_1. \end{cases}$$

We have $j(k) \to \infty$ as $k \to \infty$: If $j(k) < M$ for some M, for infinitely many k, $a_k \notin K_{j(k)}$ for all large k. Then $a_k \in K_{j(k)+1} \subseteq K_M$ for infinitely many K, which cannot occur since A has no limit points in Ω.

We claim that for any k large enough, there exists $f \in \mathrm{Hol}(\Omega)$ such that $f_k^{-1}(\{0\}) = \{a_k\}$, the multiplicity of a_k is n_k, and such that there is a holomorphic branch g_k of $\log(f_k)$ in a neighborhood of $K_{j(k)}$. We have $a_k \notin K_{j(k)}$, so $|a_k| j(k)$ or $\mathrm{dist}(a_k, \mathbb{C} \setminus \Omega) < 1/j(k)$. We deal with each case:

1. $|a_k| > j(k)$: Take $f_k(z) = (z - a_k)^{n_k}$ and then take a holomorphic branch L_k of $\log(z - a_k)$ in $\mathbb{C} \setminus \{t a_k, t \geq 1\} \supseteq K_{j(k)}$. Then $g_k = n_k L_k$.

2. $\mathrm{dist}(a_k, \mathbb{C} \setminus \Omega) < 1/j(k)$: This distance is $\int_{z \in \mathbb{C} \setminus \Omega} |a_k - z|$, and pick $b_k \in \mathbb{C} \setminus \Omega$ such that $\mathrm{dist}(a_k, \mathbb{C} \setminus \Omega) = |a_k - b_k|$. This is the infimum of a continuous function over a closed set, and it goes to ∞ as $|z| \to \infty$, so the value is achieved; moreover, $b_k \in \partial\Omega$. Take
$$f_k(z) = \left(\frac{z - a_k}{z - b_k}\right)^{n_k} \in \mathrm{Hol}(\Omega).$$
Then $\{t a_k + (1 - t) b_k : 0 \leq t \leq 1\} \cap K_{j(k)} = \varnothing$ because $\mathrm{dist}(t a_k + (1 - t) b_k, \mathbb{C} \setminus \Omega) \leq t|a_k - b_k| < 1/j(k)$. Now the Möbius transformation
$$T(z) = \frac{z - a_k}{z - b_k}$$

maps $\mathbb{C} \setminus [a_k, b_k]$ to $\mathbb{C} \setminus \overline{R}_-$, and thus we can take $g_k(z) = n_k \operatorname{Log}(T_k(z))$, where this is the principal branch of T_k. So g_k is holomorphic in a neighborhood of $K_{j(k)}$.

This proves the claim.

Now any bounded component of $\mathbb{C} \setminus K_{j(k)}$ meets $\mathbb{C} \setminus \Omega$, so by Runge's theorrem, for any k, there is a holomorphic function $h_k \in \operatorname{Hol}(\Omega)$ such that $|g_k - h_k| \leq 2^{-k}$ on $K_{j(k)}$. Define $\tilde{f}_k := e^{-h_k} f_k \in \operatorname{Hol}(\Omega)$. Then $\tilde{f}_k$ does not vanish on $K_{j(k)}$. On $K_{j(k)}$, $\tilde{f}_k = e^{g_k - h_k}$, so (using $|e^z - 1| \leq |z|e^{|z|}$) we get $|\tilde{f}_k - 1| \leq 2^{-k}e$ on $K_{j(k)}$. If $K \subseteq \Omega$, then $K \subseteq K_{j(k)}$ for large k (as $j(k) \to \infty$ when $k \to \infty$), and this estimate shows that the infinite product

$$f = \prod_{k=1}^{\infty} f_k$$

converges locally uniformly and defines $f \in \operatorname{Hol}(\Omega)$ which solves the problem. $\square$

14.2 Characterization of meromorphic functions

Weierstrass's theorem gives us an immediate way to characterize meromorphic functions.

Corollary 14.1. *Let g be meromorphic in Ω. Then $g = f/h$, where $f, h \in \operatorname{Hol}(\Omega)$.*

Proof. Let $h \in \operatorname{Hol}(\Omega)$ be such that the set of zeros of h agrees with the set of poles of g, with multiplicities. Then $f := gh \in \operatorname{Hol}(\Omega)$. $\square$

15 Corollaries of Weierstrass's Theorem and Entire Functions of Finite Order

15.1 Existence of a holomorphic function with given Taylor expansion near infinitely many points

Last time, we proved Weierstrass's theorem, which says that if $A \subseteq \Omega$ is a set with no limit points, then we can construct $f \in \mathrm{Hol}(\Omega)$ with zero set A (with multiplicities).

Proposition 15.1. *Let $\Omega \subseteq \mathbb{C}$ be open, and let $A = \{\alpha_j\}_{j=1}^\infty$ be an infinte set with no limit points in Ω. For each $j \geq 1$, let $m_j \geq 0$ be an integer, and let f_j be holomorphic near α_j. Then there exists some $f \in \mathrm{Hol}(\Omega)$ such that for all j, $f(z) - f_j(z)$ is $O(|z-\alpha_j|^{m_j+1})$ as $z \to \alpha_j$. (Thus, the Taylor expansion of f can be prescribed up to order m at each α_j.)*

Proof. By Weierstrass's theorem, we can construct $g \in \mathrm{Hol}(\Omega)$ have zeros of order $m_j + 1$ at α_j for all j. By Mittag-Leffler's theorem, there exists a meromorphic function h in Ω with poles at $\{\alpha_j\}$ only such that $h - f_j/g = r_j$ is holomorphic near α_j for all j. Define $f = gh \in \mathrm{Hol}(\Omega \setminus A)$. Then $f/g - f_j/g$ is holomorphic near α_j, so $f - f_j$ is holomorphic near α_j. So $f \in \mathrm{Hol}(\Omega)$. Also, $f - f_j = r_j g$, where r_j is $O(1)$ and g is $O(|z-a_j|^{m_j+1})$ as $z \to \alpha_j$. $\square$

15.2 Existence of a holomorphic function which cannot be extended

Here is another corollary of Weierstrass's theorem.

Corollary 15.1. *Let Ω be open. There exists $f \in \mathrm{Hol}(\Omega)$ which cannot be continued analytically to any larger open set. More precisely, if $a \in \Omega$, $g \in \mathrm{Hol}(D(a,r))$, and $f = g$ near a, then $D(a,r) \subseteq \Omega$.*

We say that Ω is the **natural domain of holomorphy** for f.

Proof. Let $\{\alpha_k\}_{k=1}^\infty$ be an enumeration of all points in Ω with rational coordinates. Let $(z_j)_{j=1}^\infty$ be a sequence in Ω such that each α_k such that each α_k occurs an infinite number of times: $(\alpha_1, \alpha_1, \alpha_1, \alpha_2, \alpha_1, \alpha_2, \alpha_3, \alpha_1, \dots)$. Choose a compact exhaustion of Ω: $K_j \subseteq \Omega$ with $K_j \subseteq K_{j+1}^o$ and $\bigcup_j K_j = \Omega$. Let $r_j = \mathrm{dist}(z_j, \mathbb{C} \setminus \Omega)$ so that $D(z_j, r_j)$ is the largest open disc centered at z_j contained in Ω. For each j, let $w_j \in D(z_j, r_j) \setminus K_j$. We let $A = \{w_j\}$; each compact set is contained in K_j for some j, so A has no limit points in Ω. Thus there exists $f \in \mathrm{Hol}(\Omega)$ such that $f^{-1}(\{0\}) = A$. Now let $a \in \Omega$ have rational coordinates and consider $D(a,r)$, where $r = \mathrm{dist}(a, \mathbb{C} \setminus \Omega)$. We have: $a = z_j$ for infinitely many j, so $D(a,r)$ contains infinitely many points w_j. Thus, by the uniqueness of analytic continuation, no function which is equal to f near a can be holomorphic in any larger disc centered at a. $\square$

Remark 15.1. When $n > 1$, this property does not hold for functions in $\mathbb{C}^n$.

15.3 Entire functions of finite order

Definition 15.1. We that $f \in \text{Hom}(\mathbb{C})$ is of **finite order** if there is some $\sigma \in \mathbb{R}$ such that $|f(z)| \le Ce^{|z|^\sigma}$ for all $z \in \mathbb{C}$ for some $C > 0$. The **order** ρ of f is the infimum of such σ.

Observe that $\rho \in [0, \infty)$. Also, f has order ρ iff for all $\varepsilon > 0$, $f(z)/e^{|z|^{\rho+\varepsilon}}$ is bounded on $\mathbb{C}$ and $f(z)/e^{|z|^{\rho-\varepsilon}}$ is unbounded on $\mathbb{C}$.

Example 15.1. Polynomials have order 0.

Example 15.2. e^z, $\cos(z)$, and $\sin(z)$ all have order 1. The function ze^z still has order 1. The function e^{z^m} has order m.

Example 15.3. The order need not be an integer. For example, $\cos(\sqrt{z})$ (defined by its Taylor expansion) has order $1/2$.

Example 15.4. Let $f \in L^1(\mathbb{R})$ be compactly supported; that is, there exists some R such that $f(x) = 0$ for a.e. x with $|x| > R$. Then the **Fourier transform** of f,

$$\hat{f}(\xi) = \int e^{-ix\xi} f(x)\, dx$$

for $\xi \in \mathbb{R}$, can be extended to the entire function

$$\hat{f}(\zeta) = \int e^{-ix\zeta} f(x)\, dx$$

for $\zeta \in \mathbb{C}$. Then

$$|\hat{f}(\zeta)| \le \int_{-R}^{R} e^{x\,\text{Im}(\zeta)} |f(x)|\, dx \le e^{R|\zeta|} \|f\|_{L^1},$$

so $\hat{f}$ is of order ≤ 1.

Remark 15.2. Let $M(r) = \max_{|z|=r} |f(z)|$. We have

$$\rho = \limsup_{r \to \infty} \frac{\log(\log(M(r)))}{\log(r)} = \lim_{R \to \infty} \left(\sup_{r \ge R} \frac{\log(\log(M(r)))}{\log(r)} \right).$$

16 Jensen's Formula

16.1 Example of entire functions of finite order

Last time, we talked about entire holomorphic functions of finite order ($|f(z)| \le Ce^{|z|^\sigma}$ for some $\sigma \in \mathbb{R}$).

Proposition 16.1. *Let f be entire of finite order ρ which is nonvanishing. Then $f = e^g$, where g is a polynomial of degree ρ.*

Proof. Write $f = e^g$, where g is entire. For any $\varepsilon > 0$, there exists a constant C_ε such that

$$|f(x)| \le C_\varepsilon e^{|z|^{\rho+\varepsilon}}.$$

So $\mathrm{Re}(g(z)) \le |z|^{\rho+\varepsilon} + \tilde{C}_\varepsilon$. By the Borel-Carathéodory inequality (proved in homework), g is a polynomial of degree $\le \rho$. As f has order ρ, we get $\deg(g) = \rho$. $\square$

16.2 Jensen's formula

Theorem 16.1 (Jensen's formula)**.** *Let $f \in \mathrm{Hol}(|z| < R)$, and assume that $f(0) \neq 0$. Let $0 < r < R$, and let $z_1, \dots, z_n$ be the zeros of f in the disc $|z| < r$, each zero repeated according to its multiplicity. Set $r_j = |z_j|$ for each $1 \le j \le n$. Then*

$$\frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\varphi})| \, d\varphi = \log\left(\frac{|f(0)| r^n}{r_1 \cdots r_n}\right).$$

If f has no zeros, this integral equals $\log|f(0)|$.

Proof. Replacing $f(z))$ by $f(rz)$, we can assume that $r = 1$. Split into cases of increasing generality:

1. $f \neq 0$ on $|z| \le 1$: Then $\log|f|$ is harmonic in a neighborhood of $|z| \le 1$, and Jensen's formula follows from the mean value property.

2. $f \neq 0$ on $|z| = 1$: Let
$$B_j(z) = \frac{\overline{z_j}(z - z_j)}{r_j(\overline{z}_j z - 1)}.$$
This is called a **Blaschke factor**. Then B_j is holomorphic near $|z| \le 1$. B_j has a simple zero atz_j only, and $|B_j(z)| \le 1$ when $|z| = 1$. Define $g = f/(B_1 \cdots B_n)$; g is holomorphic near $|z| \le 1$, nonvanishing, and $|g| = |f|$ when $|z| = 1$. Apply the previous step to g to get
$$\frac{1}{2\pi} \int_0^{2\pi} \log|f(e^{i\varphi})| \, d\varphi = \log|g(0)| = \log\left(\frac{|f(0)|}{r_1 \cdots r_n}\right).$$

3. f has (finitely many) zeros on $|z| = 1$: Apply Jensen's formula to $|z| < r$, where $r < 1$ is close to 1:

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\varphi})|\,d\varphi = \log|g(0)| = \log\left(\frac{|f(0)|r^n}{r_1\cdots r_n}\right).$$

Let $r \to 1$, and pass to the limit using dominated convergence. If $f(e^{i\varphi_0}) = 0$, estimate $|\log|f(re^{i\varphi})|$ as $r \to 1$ and $|\varphi - \varphi_0|$ is small: $f(z) = (z - e^{i\varphi_0})^m g(z)$, where g is non-vanishing. We need to consider only $|\log|r - e^{i\psi}||$ as $r \to 1$ and ψ is near 0. We get that $|\log|r - e^{i\psi}|| \le C(1 + \log(1/|\psi|))$. In particular,

$$|r - e^{i\psi}|^2 = r^2 + 1 - 2r\cos(\psi) = r\psi^2 + O(\psi^4),$$

where we have used $\cos(\psi) = 1 - \psi^2/2 + O(\psi^4)$. Altogether, if $\varphi_1, \cdots \varphi_k$ are the arguments of the zeros of f along the circle $|z| = 1$, we get:

$$|\log|f(re^{i\varphi})| \le C\left(1 + \sum_{j=1}^{k} \log_+\left(\frac{1}{|\varphi - \varphi_j|}\right)\right) \in L^1,$$

where $\log_+(t) = \max(\log(t), 0)$. So we can indeed apply the dominated convergence theorem to get Jensen's formula. □

16.3 Number of zeros in a disc

Corollary 16.1. *Let $f \in \mathrm{Hol}(|z| < R)$, and let $n = n(r)$ be the number of zeros of f in $|z| < r$, counted with multiplicities. Let the zeros be $z_1, \ldots, z_{n(r)}$ with $r_j = |z_j|$. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\varphi})|\,d\varphi - \log|f(0)| = \int_0^r \frac{n(t)}{t}\,dt.$$

Proof. Rewrite Jensen's formula using the following computation:

$$\begin{aligned}
\log\left(\frac{r^n}{r_1\cdots r_n}\right) &= \sum_{j=1}^{n}\int_{r_j}^{r} \frac{1}{t}\,dt \\
&= \sum_{j=1}^{n}\int_0^r \frac{\mathbb{1}_{(r_j,\infty)}(r)}{t}\,dt \\
&= \int_0^r \frac{1}{t}\underbrace{\left(\sum_{j=1}^{n} \mathbb{1}_{(r_j,\infty)}(r)\right)}_{=n(t)}\,dt \\
&= \int_0^r \frac{n(t)}{t}\,dt.
\end{aligned}$$

□

Remark 16.1. In particular,

$$\int_0^r \frac{n(t)}{t}\,dt \geq \int_{r/2}^r \frac{n(t)}{t}\,dt \geq n(r/2)\log(2).$$

Next time, we will use Jensen's formula to prove the following fact about entire functions of finite order.

Theorem 16.2. *Let f be entire of finite order ρ, and let $n(r) = |\{z : |z| < r.f(z) = 0\}|$. Then for all $\varepsilon > 0$ and $r \geq 1$,*

$$n(r) \leq C_\varepsilon r^{\rho+\varepsilon}.$$

17 Factorization of Entire Functions of Finite Order

17.1 Number of zeros of entire functions of finite order

Last time, we proved Jensen's formula.

Theorem 17.1. *Let f be entire of finite order ρ, and let $n(r) = |\{z : |z| < r, f(z) = 0\}|$. Then for all $\varepsilon > 0$ there exists a constant C_ε such that*

$$n(r) \leq C_\varepsilon r^{\rho+\varepsilon}$$

for all $r \geq 1$.

Proof. If $f(0) \neq 0$, then

$$\int_0^{2r} \frac{n(t)}{t}\, dt \geq \int_r^{2r} \frac{n(t)}{t}\, dt = n(r)\log(2),$$

where the inequality comes from the fact that n is increasing. Using Jensen's formula,

$$\log(2)n(r) \leq \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\varphi})|\, d\varphi + C \leq C_\varepsilon + Cr^{\rho+\varepsilon} + C \leq C_\varepsilon r^{\rho+\varepsilon}.$$

If $f(0) = 0$, apply the previous argument to $g(z) = f(z)/z^m$, where m is the multiplicity of 0. Since $n(r) = n_g(r) + m$, we get the result. ☐

17.2 Weierstrass factors and Weierstrass' theorem for $\mathbb{C}$

Definition 17.1. When $m \geq 0$ is an integer, we define the **Weierstrass factors**[3] as

$$E_m(z) = (1-z)e^{\sum_{i=1}^m z^j/j}.$$

Remark 17.1. We would like to consider infinite products of the form

$$\prod (1 - z/a_k)e^{-g(z/a_k)},$$

where $|a_k| \to \infty$ and where g should approximate $\log(1-z) = -\sum_{j=1}^\infty z^j/j$ for $|z| < 1$. The idea of the Weierstrass factors is that the factors are the partial sums of this approximation.

Lemma 17.1. *For all $|z| < 1$,*

$$|1 - E_m(z)| \leq |z|^{m+1}.$$

[3] Weierstrass used these in his proof of Weierstrass' theorem. We did not.

Proof. Let $h(z) = 1 - E_m(z)$, so $h(0) = 0$. Compute

$$h'(z) = e^{\sum_{j=1}^m z^j/j}(1 + z\varphi'(z) - \varphi'(z))' = z^m e^{\sum_{j=1}^m z^j/j}.$$

So $h(z) = O(|z|^{m+1})$, and we see that $h(z)/z^{m+1}$ is holomorphic on $\mathbb{C}$. We have

$$h'(z) = z^m(1 + a_1 z + a_2 z^2 + \cdots)$$

with $a_j \geq 0$ for all j. Integrating, we get

$$h(z) = z^{m+1}(b_0 + b_1 z + b_2 z^2 + \cdots),$$

with $b_j \geq 0$ for all j. If we write $g(z) = h(z)/z^{m+1}$, then

$$|g(z)| \leq g(|z|) \leq g(1) = h(1) = 1. \qquad \square$$

Theorem 17.2 (Weierstrass' theorem for $\mathbb{C}$)**.** *Let $(a_k)_{k=1}^\infty$ be a sequence in $\mathbb{C} \setminus \{0\}$ such that $|a_k| \to \infty$ as $k \to \infty$. Then the canonical product*

$$f(z) = \prod_{k=1}^\infty E_k(z/a_k)$$

converges locally uniformly in $\mathbb{C}$ and defines an entire function f such that $f^{-1}(\{0\}) = \{a_k\}$ and the multiplicity of $a \in f^{-1}(\{0\})$ is the number of k such that $a = a_k$.

Proof. It suffices to check that for any compact set $K \subseteq \mathbb{C}$,

$$\sum_{k=1}^\infty \sup_K |1 - E_k(z/a_k)| < \infty.$$

$K \subseteq \{|z| \leq |a_k|/2\}$ for all k large enough, and by the lemma,

$$|a - E_k(z/a_k)| \leq |z/a_k|^{k+1} \leq 2^{-k}.$$

The result follows. $\square$

17.3 Factorization of entire functions of finite order

Now assume that f is entire of finite order ρ with the zeros $a_k \neq 0$ counted with multiplicities such that $|a_1| \leq |a_2| \leq \cdots$ and $|a_k| \to \infty$.

Proposition 17.1. *The series*

$$\sum_{k=1}^\infty \frac{1}{|a_k|^{m+1}} < \infty.$$

provided that $m > \rho - 1$.

Proof. Write

$$\sum_{|a_k|\geq 1} |a_k|^{-m-1} = \sum_{j=0}^{\infty} \underbrace{\left(\sum_{2^j \leq |a_k| \leq a^{j+1}} |a_k|^{-m-1} \right)}_{2^{-j(m+1)} n(2^{j+1})}$$

$$\leq \sum_{j=0}^{\infty} C_\varepsilon 2^{(j+1)(\rho+\varepsilon)} 2^{-j(m+1)}$$

$$\leq C_\varepsilon \sum_{j=0}^{\infty} 2^{j(\rho+\varepsilon-m-1} < \infty$$

if $\rho + \varepsilon < m + 1$. □

Proposition 17.2. *Let m be the smallest integer such that $m > \rho - 1$ (so that $m \leq \rho < m + 1$). The canonical product*

$$\prod_{k=1}^{\infty} E_m(z/a_k)$$

converges locally uniformly in $\mathbb{C}$.

Remark 17.2. The improvement here is that we can use a fixed Weierstrass factor here instead of having it depend on k.

Proof. If $|z| < a_k/2$, then $|1 - E_m(z/a_k)| \leq |z/a_k|^{m+1}$. So for compact $K \subseteq \mathbb{C}$,

$$\sum \sup_K |1 - E_m(z/a_k)| < \infty.$$ □

To summarize, we can write:

$$f(z) = e^{g(z)} z^p \prod_{k=1}^{\infty} E_m(z/a_k),$$

where o is the multiplicity of 0 as the zero of f, and g is entire. This will allow us to understand the structure of entire functions of finite order in the following way:

Theorem 17.3 (Hadamard)**.** *The function g is a polynomial of degree $\leq \rho$.*

18 Hadamard Factorization

18.1 Lower bound on the product of Weierstrass factors

Let f be entire of finite order ρ, with zeros (a_k) such that $0 < |a_1| \leq a_2| \leq \cdots$. Let $m \in \mathbb{N}$ be such that $m \leq \rho < m+1$. Then we have the **Hadamard factorization**:

$$f(z) = e^{g(z)} z^p \prod_{k=1}^{\infty} E_m(z/a_k),$$

where g is entire, and p is the order of the zero at $z = 0$.

Theorem 18.1 (Hadamard)**.** *The function g is a polynomial of degree $\leq p$.*

We need a good lower bound on the canonical product away from the zeros $\{a_k\}$.

Proposition 18.1. *For any $s \in \mathbb{R}$ such that $\rho < s < m+1$, there is a constant $C_s = C > 0$ such that*

$$\left| \prod_{k=1}^{\infty} E_m(z/a_k) \right| \geq e^{-C|z|^s}$$

for all $z \in \mathbb{C} \setminus \bigcup D(a_k, |a_k|^{-m-1})$.

Proof. We need the following 2 estimates for $E_m(z)$:

1. $|E_m(z)| \geq e^{-C|z|^{m+1}}$ when $|z| < 1/2$: Write

   $$E_m(z) = (1-z)e^{\sum_{j=1}^m z/j} = e^w,$$

   where

   $$w = \log(1-z) + \sum_{j=1}^{m} \frac{z^j}{j} = -\sum_{j=m+1}^{\infty} \frac{z^j}{j}.$$

   So $|w| \leq 2|z|^{m+1}$, and the estimate follows.

2. $E_m(z)| \geq |1-z|e^{-C|z|^m}$ when $|z| > 1/2$: Write

   $$|E_m(z)| \geq |1-z|e^{-|\sum_{j=1}^m z^j/j|},$$

   where

   $$\left| \sum_{j=1}^{m} \frac{z^j}{j} \right| \leq |z|^m \sum_{j=1}^{m} \frac{1}{|z|^{m-j}} \leq C|z|^m.$$

We write next

$$\prod_{j=1}^{\infty} E_m(z/a_k) = \underbrace{\prod_{|z/a_k|<1/2} E_m(z/a_k)}_{=A} \underbrace{\prod_{|z/a_k|\geq 1/2} E_m(z/a_k)}_{=B}.$$

The first estimate gives

$$|A| \geq \prod_{|z/a_k|<1/2} e^{-C|z/a_k|^{m+1}} = e^{-C|z|^{m+1}\sum_{|a_k|>2|z|} 1/|a_k|^{m+1}}.$$

Now if $\rho < s < m+1$, then $\sum 1/|a_k|^s < \infty$ (by the same argument as in last lecture). Then $|a_k|^{-m-1} = |a_k|^{-s}|a_k|^{s-m-1} \leq C|a_k|^{-s}|z|^{s-m-1}$, so we get the lower bound

$$|A| \geq e^{-C_s|z|^s}.$$

Next, the second estimate gives

$$|B| \geq \prod_{|z/a_k|>1/2} |1 - z/a_k| \underbrace{\prod_{|z/a_k|\geq 1/2} e^{-C|z/a_k|^m}}_{=\exp(-C|z|^m \sum 1/|a_k|^m)}.$$

To bound this second term, we have $|a_k|^{-m} = |a_k|^{-s}|a_k^{s-m} \leq C|z|^{s-m}|a_k|^{-s}$, so

$$\prod_{|z/a_k|\geq 1/2} e^{-C|z/a_k|^m} \geq e^{-C_s|z|^s}.$$

Finally, using $|z - a_k| \geq 1/|a_k|^{m+1}$ for all k, we get

$$\prod_{|z/a_k|\geq 1/2} |1 - a/z_k| \geq \prod_{|z/a_k|\geq 1/2} \frac{1}{|a_k|^{m+2}}.$$

Taking logs, we get

$$\sum_{|a_k|\leq 2|z|} (m+2)\log|a_k| \leq O(1)\log(2|z|) \underbrace{n(2|z|)}_{\leq C_\varepsilon |z|^{\rho+\varepsilon}} \leq O(1)|z|^s.$$

The result follows. □

18.2 Proof of Hadamard's theorem

Let $\Omega = \mathbb{C} \setminus \bigcup D(a_k, 1/|a_k|^{m+1})$ be the domain from the previous proposition.

Proposition 18.2. *There exists a sequence $R_k \to \infty$ such that $\{|z| = R_k\} \subseteq \Omega$.*

Proof. Recall that $\sum_{k=1}^\infty 1/|a_k|^{m+1} < \infty$. Pick N so that $\sum_{k=N}^\infty 1/|a_k|^{m+1} < 1/2$. Set $A_k = \{x \in \mathbb{R} : |x - |a_k|| \leq |a_k|^{-m-1}\}$. Then $\sum_{k=N}^\infty < 1$. Given $L \in \mathbb{N}$ large, let $r \in [L_1, L+1] \setminus \bigcup_{k=N}^\infty A_k$; the set $\bigcup_{k=N}^\infty A_k$ has Lebesgue measure < 1. Then if $|z| = r$,

$$|z - a_k| \geq ||z| - |a_k|| \geq \frac{1}{|a_k|^{m+1}}.$$

If $L \geq L_0$ for large L_0, we also get

$$|z - a_k| \geq \frac{1}{|a_k|^{m+1}}$$

for $1 \leq k \leq N$, and the result follows. $\square$

Now we can prove Hadamard's theorem. Recall that we have

$$f(z) = e^{g(z)} z^p \prod_{k=1}^\infty E_m(z/a_k).$$

Proof. When $|z| = R_j$, we have

$$|e^{g(z)}| = \frac{|f(z)|}{|z^p| \underbrace{\prod E_m(z/a_k)}_{\geq C_\varepsilon \exp(-|z|^{\rho+\varepsilon})}} \leq C_\varepsilon e^{|z|^{\rho+\varepsilon}}$$

for al $\varepsilon > 0$. By the Borel-Carathéodory estimate, which says

$$\sup_{|z|=r} |g(z)| \leq \frac{2r}{R-r} \sup_{|z|=R} \operatorname{Re}(g(z)) + \frac{R+r}{R-r}|g(0)|, \qquad r < R,$$

there exists a sequence $R_j \to \infty$ such that

$$|g(z)| \leq C_\varepsilon + |z|^{\rho+\varepsilon}, \qquad |z| = R_j, j = 1, 2, \ldots$$

By the usual Cauchy's estimates argument, g is a polynomial of degree $\leq \rho$. $\square$

19 Applications of Hadamard Factorization and Properties of the Γ-Function

19.1 Minimum modulus theorem and range of entire functions of finite order

Last time, we proved the Hadamard factorization for entire functions of finite order:

$$f(z) = e^{g(z)} z^p \prod_{k-1}^{\infty} E_m(z/a_k),$$

where (a_k) are the zeros of f such that $0 < |a_1| \leq |a_2| \leq \cdots$, p is the order of the zeros at 0, $m \leq \rho < m+1$, and g is a polynomial of degree $\leq \rho$. We have for all $s \in (\rho, m+1)$ there exists some $C > 0$ such that

$$\left| \prod E_m(z/a_k) \right| \geq e^{-C|z|^s}, \qquad z \in \mathbb{C} \setminus \bigcup D(a_k, 1/|a_k|^{m+1}).$$

Our analysis of this gives us the following facts:

Corollary 19.1 (minimum modulus theorem)**.** *For every $\varepsilon > 0$, there exists an $R > 0$ such that*

$$|f(z)| \geq e^{-|z|^{\rho+\varepsilon}}, \qquad |z| \geq R, \quad z \in \mathbb{C} \setminus \bigcup D(a_k, 1/|a_k|^{m+1}).$$

Corollary 19.2. *Let f be entire of finite order $\rho \notin \mathbb{N}$. Then f assumes every complex value infinitely many times.*

Proof. For any $w \in \mathbb{C}$, $f, f-w$ are entire of the same order, so it suffices to show that f has infinitely many zeros. If f has only finitely many zeros, then the Hadamard factorization gives $f(z) = p(z)e^{g(z)}$, where p, g are polynomials. The order of such a function is the degree of g, which is an integer. □

19.2 Factorization of sine

Example 19.1. Let $f(z) = \sin(\pi z)$. This is entire of order 1, and $f^{-1}(\{0\}) = \mathbb{Z}$. Write $\mathbb{Z} \setminus \{0\}$ as $\{a_k : k = 1, 2, \dots\}$ with $a_{2j} = -j$ for $j \geq 1$ and $a_{2j+1} = j+1$, for $j \geq 0$. We can write

$$\begin{aligned}\sin(\pi z) &= e^{g(z)} z \prod_{k=1}^{\infty} E_1(z/a_k) \\ &= e^{g(z)} z \prod_{k=1}^{\infty} (1 - z/a_k) e^{z/a_k}\end{aligned}$$

$$= e^{g(z)} z \prod_{j=1}^{\infty} (1 + z/j) e^{-z/j} \prod_{j=0}^{\infty} (1 - z/(j+1)) e^{z/(j+1)}$$
$$= e^{g(z)} z \prod_{j=1}^{\infty} (1 + z^2/j^2)$$

e^g is even, and g is a polynomial of degree ≤ 1. So $g(z) = g(= z) + 2\pi k i$ for some $k \in \mathbb{Z}$. If $g(z) = \alpha z + \beta$, then $\alpha = 0$.

$$= e^{\beta} z \prod_{j=1}^{\infty} (1 + z^2/j^2).$$

To find β, differentiate and take $z = 0$ to get $\pi = e^{\beta}$. This gives us the classical factorization formula:

$$\sin(\pi z) = \pi z \prod_{j=1}^{\infty} (1 - z^2/j^2).$$

19.3 The Γ-function

Definition 19.1. The **Γ-function** is defined by

$$\Gamma(a) = \int_0^{\infty} t^{a-1} e^{-t} \, dt, \qquad \mathrm{Re}(a) > 0.$$

The integral converges locally uniformly in $\mathrm{Re}(a) > 0$ and defines a holomorphic function in this region. We have

$$\Gamma(a+1) = \lim_{\substack{\varepsilon \to 0^+ \\ R \to \infty}} \int_{\varepsilon}^{R} e^{-t} t^a \, dt = \lim_{\substack{\varepsilon \to 0^+ \\ R \to \infty}} \left(-t^a e^{-t} \Big|_{\varepsilon}^{R} + \int_{\varepsilon}^{R} a t^{a-1} e^{-t} \, dt \right) = a\Gamma(a),$$

when $\mathrm{Re}(a) > 0$. In particular, since $\Gamma(1) = 1$, we have

$$\Gamma(n) = (n-1)!, \qquad n \geq 1.$$

Proposition 19.1. *The Γ-function has a meromorphic continuation to $\mathbb{C}$ with simple poles at the nonpositive integers $\{0, -1, -2, \dots\}$. The residue at $-N$ is $(-1)^N/N!$.*

Proof. For $N \in \mathbb{N}$ with $N > 0$, write

$$\begin{aligned} \Gamma(a+N+1) &= (a+N)\Gamma(a+N) \\ &= (a+N)(a+N-1)\Gamma(a+N-1) \\ &= \cdots \end{aligned}$$

$$= (a+N)\cdots(a+1)a\Gamma(a).$$

So we can write

$$\Gamma(a) = \frac{\Gamma(a+N+1)}{(a+N)\cdots(a+1)a}.$$

The right hand side is meromorphic in $\mathrm{Re}(a) > -N-1$. Thus, Γ extends meromorphically to all of $\mathbb{C}$ with the poles $\{0,-1,-2,\dots\}$. Compute

$$\mathrm{Res}(\Gamma,-N) = \lim_{a\to -N}(a+N)\Gamma(a) = \frac{(-1)^N}{N!} \qquad \square$$

Remark 19.1. We have $\Gamma(a+1) = a\Gamma(a)$ for $a \in \mathbb{C}\setminus\{0,-1,-2,\dots\}$.

We want to apply Hadamard factorization to Γ, but it is not entire. However, $1/\Gamma$ is entire. We will use the following property of the Γ function:

Proposition 19.2 (reflection identity)**.** *For $a \in \mathbb{C}\setminus\mathbb{Z}$,*

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}$$

Proof. It suffices to show the identity when $0 < \mathrm{Re}(a) < 1$. Write

$$\Gamma(1-a) = \int_0^\infty e^{-x}x^-a\,dx \overset{x=ty}{=\!=} t\int_0^\infty e^{-ty}(ty)^{-a}\,dy.$$

so we may write

$$\begin{aligned}\Gamma(a)\Gamma(1-a) &= \int_0^\infty e^{-t}t^{a-1}t\left(\int_0^\infty e^{-ty}(ty)^{-a}\,dy\right)dt \quad = \iint_{t\geq 0, y\geq 0} e^{-t(1+y)}y^{-a}\,dy\,dt\\ &= \int_0^\infty \frac{y^{-a}}{1+y}\,dy\\ &= \frac{\pi}{\sin(\pi a)}.\end{aligned}$$

To show the last equality apply the residue theorem to

$$f(z) = \frac{z^{b-a}}{1+z}$$

with $0 < b < 1$ and $0 < \arg(z) < 2\pi$, using a "keyhole contour." We get

$$\int_\gamma f(z),\ dz \to (1-e^{2\pi i(b-1)})\textstyle\int_0^\infty \displaystyle\frac{x^{b-1}}{1+x}\,dx,$$

where the left hand side equals $2\pi i(-1)^{b-1}$. $\square$

Next time, we will show that $1/\Gamma$ is entire of order 1.

20 Uniqueness of the Γ-Function and Hadamard Factorization of $1/\Gamma$

20.1 Uniqueness of the Γ-function

Last time, we defined the Γ-function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

We saw that $\Gamma \in \mathrm{Hol}(\mathrm{Re}(z) > 0)$ and extends meromorphically to all of $\mathbb{C}$ with simple poles at $\{0, -1, -2, \dots\}$. We also saw that

$$\Gamma(z+1) = z\Gamma(z),$$
$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

the latter of which is called the "reflection identity."

The functional equation actually characterizes Γ.

Proposition 20.1. *Let $f \in \mathrm{Hol}(\mathrm{Re}(z) > 0)$ be such that $f(z+1) = zf(z)$, and assume that f is bounded in $1 \leq \mathrm{Re} \leq 2$. Then $f(z) = f(1)\Gamma(z)$.*

Proof. Consider $\tilde{f}(z) = f(z) - f(1)\Gamma(z)$. We have $\tilde{f}(z+1) = z\tilde{f}(z)$, so $\tilde{f}$ extends meromorphically to $\mathbb{C}$ with simple poles at $\{0, -1, -2, \dots\}$, and we can write

$$\tilde{f}(z) = \frac{\tilde{f}(z+N-1)}{z(z+1)\cdots(z+N)}, \qquad \mathrm{Re}(z) > -N-1.$$

So $\mathrm{Res}(\tilde{f}, -N) = \lim_{z \to -N}(z+N)\tilde{f}(z) = 0$ for all N. So $\tilde{f}$ is entire. Set $\tilde{u}(z) = \tilde{f}(z) = \tilde{f}(z)\tilde{f}(1-z) \in \mathrm{Hol}(\mathbb{C})$, and we get

$$\tilde{u}(z+1) = \tilde{f}(z+1)\tilde{f}(-z) = z\tilde{f}(z)\frac{1}{-z}\tilde{f}(1-z) = -\tilde{u}(z).$$

So $\tilde{u}$ is antiperiodic and bounded in $1 \leq \mathrm{Re}(z) \leq 2$, so $\tilde{u}$ is constant. So we get $\tilde{u}(z) = \tilde{u}(1) = 0$. $\square$

20.2 Hadamard factorization of $1/\Gamma$

Theorem 20.1. *The function $1/\Gamma$ is entire of finite order 1 with the Hadamard factorization*

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{k=1}^\infty (1 + z/k)e^{-z/k},$$

where $\gamma = \lim_{N\to\infty} \sum_{n=1}^N 1/n - \log(N)$ is the Euler constant.

Proof. We have the reflection identity

$$\frac{1}{\Gamma(z)} = \Gamma(1-z)\frac{\sin(\pi z)}{\pi}$$

for all $z \in \mathbb{C}$. The sine term is of order 1. We have

$$\begin{aligned}
\Gamma(z) &= \int_0^1 e^{-t}t^{z-1}\,dt + \int_1^\infty e^{-t}t^{z-1}\,dt \\
&= \sum_{j=0}^\infty \int_0^1 \frac{(-t)^j}{j!}t^{z-1}\,dt + \int_1^\infty e^{-t}t^{z-1}\,dt \\
&= \sum_{j=0}^\infty \frac{(-1)^j}{j!(j+z)} + \underbrace{\int_1^\infty e^{-t}t^{z-1}\,dt}_{\in \mathrm{Hol}(\mathbb{C})}.
\end{aligned}$$

The series defines a meromorphic function in $\mathbb{C}$ with poles at $\{0,-1,-2,\dots\}$ since for every compact set $K \subseteq \mathbb{C}$, the functions $(-1)^j/(j!(j+z))$ have no poles in K for $j \geq j_0$ and because $\sum_{j=j_0}^\infty (-1)^j/(j!(j+z))$ converges uniformly on K. We get by analytic continuation that

$$\Gamma(1-z) = \sum_{j=0}^\infty \frac{(-1)^j}{j!(j+1-z)} + \int_1^\infty e^{-t}t^{-z}$$

for any z, so

$$\frac{1}{\Gamma(z)} = \sum_{j=0}^\infty \frac{(-1)^j}{j!(j+1-z)}\frac{\sin(\pi z)}{\pi} + \left(\int_1^\infty e^{-t}t^{-z}\right)\frac{\sin(\pi z)}{\pi}.$$

Now

$$\left|\int_1^\infty e^{-t}t^{-z}\,dt\right| \leq \int_1^\infty e^{-t}e^{|\operatorname{Re}(z)|}\,dt$$

Let $|\operatorname{Re}(z)| \leq n < 1 + |\operatorname{Re}(z)|$, where $n \in \mathbb{N}$.

$$\begin{aligned}
&\leq n! \\
&\leq n^n \\
&\leq e^{(1+|z|)\log(1+|z|)},
\end{aligned}$$

so we get

$$\left|\left(\int_1^\infty e^{-t}t^{-z}\right)\frac{\sin(\pi z)}{\pi}\right| \leq Ce^{C(1+|z|)\log(1+|z|)}.$$

If $|\operatorname{Im}(z)| \geq 1$, then

$$\left| \left(\sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+1-z)} \right) \frac{\sin(\pi z)}{\pi} \right| \leq Ce^{\pi|z|}.$$

The same estimate holds if $\operatorname{Re}(z) \leq 1/2$. Let $k \in \mathbb{N}_+$ with $k \geq 1$ be such that $k - 1/2 \leq \operatorname{Re}(z) < k + 1/2$. Then

$$\left(\sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+1-z)} \right) \frac{\sin(\pi z)}{\pi} = \underbrace{\frac{(-1)^k}{k!(k-z)} \frac{\sin(\pi z)}{\pi}}_{O(1)} + O(1)e^{\pi|z|}.$$

It follows that the order of $1/\Gamma$ is ≤ 1.

To see that the order is ≥ 1, write

$$\Gamma(z) = \frac{\Gamma(z+N+1}{z(z+1)\cdots(z+N)}, \qquad \operatorname{Re}(z) > -N-1.$$

and take $z = N - 1/2$. Then

$$\left| \frac{1}{\Gamma(-N-1/2)} \right| \geq \frac{1}{N!} \geq \frac{1}{C} N^N e^{-N}$$

by Stirling's formula. So the order of $1/\Gamma$ is exactly 1.

By Hadamard's theorem, we get

$$\frac{1}{\Gamma(z)} = e^{\alpha z + \beta} z \prod_{k=1}^{\infty} (1 - z/k) e^{-z/k}.$$

Multiply both sides by $\Gamma(z)$, and let $z \to 0$. We get

$$1 = \lim_{z \to 0} e^{\alpha z + \beta} \Gamma(z) z = e^{\beta},$$

so $\beta = 0$. To compute $\alpha \in \mathbb{R}$, take $z = 1$ in the expression for $1/\Gamma$:

$$1 = \frac{1}{\Gamma(z)} e^{\alpha} \prod_{k=1}^{\infty} (1 + 1/k) e^{-1/k},$$

so

$$e^{-\alpha} = \lim_{N \to \infty} \exp\left(-\sum_{k=1}^{N} 1/k + \sum_{k=1}^{N} \log(k+1) - \log(k) \right).$$

We get that

$$\alpha = \lim_{N \to \infty} \sum_{k=1}^{N} \frac{1}{K} - \log(N). \qquad \square$$

Next, we will discuss the range of holomorphic functions with Picard's theorems.

Theorem 20.2 (Picard's little theorem)**.** *Let $f \in \mathrm{Hol}(\mathbb{C})$ be entire and nonconstant. Then the range $f(\mathbb{C})$ omits at most 1 point of $\mathbb{C}$.*

21 Bloch's Theorem and Range of Meromorphic Functions

21.1 Bloch's theorem

We want to prove the following theorem.

Theorem 21.1 (Picard's little theorem)**.** *Let $f \in \mathrm{Hol}(\mathbb{C})$ be entire and nonconstant. Then the range $f(\mathbb{C})$ omits at most 1 point of $\mathbb{C}$.*

Remark 21.1. It is possible for the range to omit one point. $e^z \neq 0$ for all $z \in \mathbb{C}$.

Remark 21.2. There exists a topological proof of this fact, but it requires the machinery of covering spaces, so we will not visit it at this time.

Proposition 21.1. *Let $f \in \mathrm{Hol}(|z| < 1)$ be such that $f(0) = 0$, $f'(0) = 1$. If, furthermore, $|f| \leq M$, then $f(\{|z| < 1\}) \supseteq D(0, 1/(4M))$.*

We will write $D := \{|z| < 1\}$.

Remark 21.3. If $M = 1$, then $f(D) = D$ by Schwarz's lemma.

Proof. Let $w \in \mathbb{C} \setminus f(D)$. Then $w \neq 0$, the function $1 - f/w \neq 0$ in D, and $1 = f/w = 1$ at $z = 0$. Then there exists $g \in \mathrm{Hol}(|z| < 1)$ such that $g^2 = 1 - f/w$ and $g(0) = 1$. Differentiate and let $z = 0$ to get $2g(0)g'(0) = -1/w$. So $g'(0) = -1/(2w)$, which gives the Taylor expansion

$$g(z) = 1 - \frac{z}{2w} + \cdots .$$

Now given $h \in \mathrm{Hol}(|z| < 1)$, we have $h = \sum_{n=0}^{\infty} a_n z^n$ and

$$\frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\varphi})|^2 \, d\varphi = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

for $r < 1$. In particular, apply this property to g. Then

$$\frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\varphi})|^2 \, d\varphi \leq \|g\|_\infty^2 \leq 1 + \frac{M}{|w|}$$

and

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \geq 1 + \frac{r^2}{4|w|^2}.$$

Sending $r \to 1$, we get $1/(4|w|^2) \leq M/|w|$. That is, $|w| \geq 1/(4M)$. $\square$

Theorem 21.2 (A. Bloch)**.** *There exists an absolute constant $\ell > 0$ such that if $f \in \mathrm{Hol}(|z| < 1)$ and $f'(0) = 1$, then the range of $f(D)$ contains an open disc of radius ℓ.*

Proof. Assume first that f is holomorphic near $|z| \le 1$. Let $\mathrm{Aut}(D)$ be the set of holomorphic bijections $\varphi : D \to D$; this is the set of automorphisms of D:

$$\varphi \in \mathrm{Aut}(D) \iff \varphi(z) = \lambda \frac{z-\alpha}{1-\overline{\alpha} z},$$

where $|\lambda| = 1$, and $\alpha \in D$. We have

$$(1-|z|^2)|\varphi'(z)| = 1 - |\varphi(z)|^2$$

for all $\varphi \in \mathrm{Aut}(D)$. Define $B(f,z) = (1-|z|^2)|f'(z)|$ when $z \in D$. For any $\varphi \in \mathrm{Aut}(D)$,

$$B(f \circ \varphi, z) = (1-|z|^2)|f'(\varphi(z))||\varphi'(z)| = (1-|\varphi(z)|)^2|f'(\varphi(z))| = B(f, \varphi(z)).$$

The function $B(f,\cdot)$is continuous in D, nonnegative, and equal to 0 on ∂D. Let $a \in D$ be such that B achieves its maximum at a.

Assume first that $a = 0$. Then $|f'(z)| \le 1/(1-|z|^2)$ for $|z| < 1$ for $|z| < 1$. We get

$$|f(z) - f(0)| = \left| \int_0^1 \frac{d}{dt} f(tz)\, dt \right| \le \frac{|z|}{1-|z|^2}, \qquad |z| < 1.$$

If $|z| \le R < 1$, we get

$$|f(z) - f(0)| \le \frac{R}{1-R^2} = M.$$

Apply the previous proposition to $(f(Rz) - f(0))/R$, which is a holomorphic function bounded by M/R. Then $f(D)$ contains an open disc of radius $R\frac{1}{4(M/R)} = R^2/(4M) = R(1-R)^2/4$. This is true for any $0 < R < 1$, so the optimal choice of R is $\sqrt{3}/3$. The corresponding radius is $\sqrt{3}/18$.

In general, we may have $a \neq 0$. Let $\psi \in \mathrm{Aut}(D)$ be such that $\psi(0) = a$. Consider $g = f \circ \psi$. Then

$$B(g,z) = B(f,\psi(z)) \le B(f,a) = B(g,0),$$

by pulling back using ψ and the conformal invariance of B. Note that the right hand side equals $|g'(0)|$, so $|g'(0)| \ge 1$. The previous discussion can be applied to the function $(g(Rz) - g(0))/(Rg'(0))$. So the $g(D)$ contains an open disc of radius $\sqrt{3}18|g'(0)| \ge \sqrt{3}18$. Since $g(D) = f(D)$. we get the result, if f is holomorphic near $|z| \le 1$.

In general, let $f_\rho(z) = (1/\rho) f(\rho z)$, where $0 < \rho < 1$. Then $f_\rho(D)$ contains a fixed disc. Then $f(D) \supseteq \rho f_\rho(D)$, which contains a disc of radius $\rho\sqrt{3}/18$. Pick any such ρ to get the theorem. $\square$

21.2 Range of meromorphic functions

We will use Bloch's theorem to prove Picard's little theorem next time. Here is a corollary of Picard's theorem.

Corollary 21.1. *Let f be meromorphic in $\mathbb{C}$ and nonconstant. Then f assumes all values in $\mathbb{C}$ with at most 2 exceptions.*

Proof. Assume f does not take on the distinct values $a, b, c \in \mathbb{C}$. Let $g(z) = 1/(f(z) - c)$. This is holomorphic away form the poles of f. The singularities at the poles of f are removable for g, so g can be extended to an entire holomorphic function. Its range omits 2 values: $1/(a - c)$ and $1/(b - c)$. So g is constant by Picard's little theorem. □

Example 21.1. Let

$$f(z) = \frac{1}{e^z + 1}.$$

This function omits the values $0, 1$.

Example 21.2. Suppose we try to solve $f^n + g^n = 1$ with $n \geq 3$. This equation has no nonconstant solution by this corollary to Picard's little theorem.

22 Picard's Little Theorem and Schottky's Theorem

22.1 Picard's little theorem

Last time, we proved Bloch's theorem:

Theorem 22.1 (A. Bloch)**.** *There exists an absolute constant $\ell > 0$ such that if $f \in \mathrm{Hol}(|z| < 1)$ and $f'(0) = 1$, then the range of $f(D)$ contains an open disc of radius ℓ.*

We can now prove prove Picard's little theorem.[4]

Theorem 22.2 (Picard's little theorem)**.** *Let $f \in \mathrm{Hol}(\mathbb{C})$ be entire and nonconstant. Then the range $f(\mathbb{C})$ omits at most 1 point of $\mathbb{C}$.*

Proof. Let $f \in \mathrm{Hol}(\mathbb{C})$, and assume that f omits 2 distinct values $a, b \in \mathbb{C}$. By composing with an affine transformation, we may assume that $a = 0$, $b = 1$. We will show that f is constant.

We claim that there exists $g \in \mathrm{Hol}(\mathbb{C})$ such that $f(z) = -\exp(i\pi \cosh(2g(z)))$. The function $f \neq 0$ in $\mathbb{C}$, so there exists $F \in \mathrm{Hol}(\mathbb{C})$ such that $e^{2\pi i F} = f$. Moreover, F does not assume integer values, so we can define $\sqrt{F} - \sqrt{F-1} \in \mathrm{Hol}(\mathbb{C})$ which is also nonvanishing. Define g as a holomorphic branch of $\log(\sqrt{F} - \sqrt{F-1})$. Then

$$\begin{aligned} e^{g} &= \sqrt{F} - \sqrt{F-1}, \\ e^{-g} &= \sqrt{F} + \sqrt{F-1} \end{aligned}$$

so

$$\cosh(2g) + 1 = 2\cos^2(g) = 2F,$$

which proves the claim.

Let

$$E = \{\pm \underbrace{\log(\sqrt{n} + \sqrt{n-1})}_{=\lambda_n} + im\pi/2 : m \in \mathbb{Z}, n \geq 1\}.$$

The points in E form the vertices of a grid of rectangles in $\mathbb{C}$. We claim that $E \cap g(\mathbb{C}) = \varnothing$. If $g(z) = \pm \log(\sqrt{n} + \sqrt{n-1}) + im\pi/2$, then

$$\begin{aligned} 2\cosh(2g(z)) &= e^{im\pi}\left((\sqrt{n} + \sqrt{n-1})^2 + (\sqrt{n} - \sqrt{n-1})^2\right)^2 \\ &= (-1)^m 2(2n-1), \end{aligned}$$

so $f(z) = 1$.

We now claim that g is constant. We have that the height of a rectangle R_n in our grid is $\pi/2$, and the width of R_n is $\lambda_{n+1} - \lambda_n = \log\left(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n}+\sqrt{n-1}}\right) \leq C$ for $n \geq 1$. So there exists some $R_0 > 0$ such that each open disc of radius R_0 meets E. If $g'(a) \neq 0$ for some a, then apply Bloch's theorem to the function $g(a + Rz)/Rg'(a)$ for $|z| < 1$, $R > 0$. The range contains a disc of fixed radius $\ell > 0$ for each $R > 0$, so $g(\mathbb{C})$ contains a disc of radius $R\ell|g'(a)|$. But $g(\mathbb{C}) \cap E = \varnothing$, so $R\ell|g'(a)| \leq R_0$; letting $R \to \infty$, we get a contradiction. □

[4] This proof is not Picard's original proof. Bloch's theorem came after the original proof.

22.2 Schottky's theorem

Here is a consequence of Bloch's theorem. It will allow us to prove Picard's great theorem.

Theorem 22.3 (Schottky)**.** *For each $0 < \alpha < \infty$ and $0 \leq \beta \leq 1$, there exists a constant $M(\alpha, \beta) > 0$ such that if $f \in \mathrm{Hol}(D)$ omits the values $0, 1$ and $|f(0)| \leq \alpha$, then $|f(z)| \leq M(\alpha, \beta)$ for all $|z| \leq \beta$.*

Proof. We may assume $\alpha \geq 2$. Assume that $1/2 \leq |f(0)| \leq \alpha$. Following the proof of Picard's little theorem, let $F \in \mathrm{Hol}(D)$ be such that $e^{2\pi i F} = f$ in D. Chose the branch of f so that $\mathrm{Re}(F(0)) \in [0, 1]$. Then $e^{-2\pi \operatorname{Im}(F(0))} = |f(0)|$, so $|\operatorname{Im}(F(0))| \leq (1/2\pi)\log(\alpha)$. We will call $C(\alpha)$ any constant that depends only on α. So $|F(0)| \leq C(\alpha)$. Next, $\sqrt{F} - \sqrt{F-1} \in \mathrm{Hol}(D)$, and $|\sqrt{F(0)} - \sqrt{F(0)-1}| \leq |F(0)|^{1/2} + (|F(0)|+1)^{1/2} \leq C(\alpha)$. Finally, let $g \in \mathrm{Hol}(D)$ be such that $e^g = \sqrt{F} - \sqrt{F-1}$. Choose the branch so that $0 \leq \operatorname{Im}(g(0)) < 2\pi$. We can then control $|\mathrm{Re}(g(0))|$. We get a constant $C(\alpha) > 0$ such that if $f(z) = \exp(i\pi\cosh(2g(z)))$, then $|g(0)| \leq C(\alpha)$ if $1/2 \leq |f(0)| \leq \alpha$.

Recall that $g(D) \cap E = \varnothing$, where E is as in the proof of Picard's little theorem. Then there is a number R_0 such that $g(D)$ contains no disc. Let $|z| \leq \beta < 1$, and let

$$\varphi(\zeta) = \frac{g(z + (1-\beta)\zeta)}{(1-\beta)g'(z)}$$

where z is such that $g'(z) \neq 0$. This is holomorphic in $|\zeta| < 1$, and $\varphi'(0) = 1$, so $\varphi(D)$ contains a disc of radius ℓ by Bloch's theorem. So $g(D)$ contains a disc of radius $|ell(1-\beta)|g'(z)|$. So $|g'(z)| \leq R_0/(\ell(1-\beta))$ for $|z| \leq \beta$. By integration, we get uniform control on the function g. $\square$

We will finish the proof next time.

23 The Montel-Caratheodory Theorem and Corollaries of Picard's Great Theorem

23.1 Proof of Schottky's theorem, continued

Last time, we were proving Schottky's theorem. Let's finish the proof.

Theorem 23.1 (Schottky)**.** *For each $0 < \alpha < \infty$ and $0 \leq \beta < 1$, there exists a constant $M(\alpha, \beta) > 0$ such that if $f \in \mathrm{Hol}(D)$ omits the values $0, 1$ and $|f(0)| \leq \alpha$, then $|f(z)| \leq M(\alpha, \beta)$ for all $|z| \leq \beta$.*

Proof. It suffices to show this for when $\alpha \geq 2$.

Case 1: $1/2 \leq |f(0)| \leq \alpha$: We have shown that we can write $f = -\exp(i\pi \cosh(2g(z)))$ with $g \in \mathrm{Hol}(D)$, $|g(0)| \leq C(\alpha)$, and $|g'(z)| \leq C_0/(1-\beta)$ for $|z| \leq \beta < 1$, for some absolute constant C_0. Writing $g(z) = g(0) = \int_0^1 zg'(tz)\, dt$, we get

$$|g(z)| \leq C(\alpha) + \frac{C_0|z|}{1-\beta} \leq C(\alpha, \beta), \qquad |z| \leq \beta < 1.$$

We get

$$|f(z)| \leq e^{\pi e^{2|g(z)|}} \leq M(\alpha, \beta).$$

Case 2: $0 < |f(0)| < 1/2$: Apply case 1 to the function $1-f$. Then $1/2 \leq |1-f(0)| \leq 2$. So, by case 1, $|1 - f(z)| \leq M(2, \beta)$ for $|z| \leq \beta < 1$. □

23.2 The Montel-Caratheodory theorem

Definition 23.1. Let $\Omega \subseteq \mathbb{C}$ be open, and let $\mathcal{F} \subseteq \mathrm{Hol}(\Omega)$. We say $\mathcal{F}$ is **normal** if each sequence in $\mathcal{F}$ has a subsequence which either converges locally uniformly in $\mathrm{Hol}(\Omega)$ or tends to ∞ uniformly on each compact set.

Theorem 23.2 (Montel-Caratheodory)**.** *Let $\Omega \subseteq \mathbb{C}$ be open and connected, and let $\mathcal{F} \subseteq \mathrm{Hol}(\Omega)$ be such that for any $f \in \mathcal{F}$, $f(\Omega)$ omits the values 0,1. Then $\mathcal{F}$ is normal.*

Proof. Let (f_n) be a sequence in $\mathcal{F}$. It suffices to show that for any open disc D with $\overline{D} \subseteq \Omega$, there exists a subsequence of (f_n) which converges uniformly on D or tends to ∞ unformly on D. Let $(D_\nu)_{\nu=1}^\infty$ be such that $\overline{D}_\nu \subseteq \Omega$, $\Omega = \bigcup_{\nu=1}^\infty D_\nu$. Passing to a suitable diagonal subsequence (g_n) of (f_n), we get that for all ν, (g_n) converges uniformly on D_ν or tends to ∞ uniformly on D_ν. Let Ω_1 be the set of $z \in \Omega$ such that (g_n) converges uniformly in a neighborhood of z, and let Ω_2 be the set of $z \in \Omega$ such that (g_n) tends to ∞ uniformly in a neighborhood of z. Then Ω_1, Ω_2 are open and disjoint, and $\Omega = \Omega_1 \cup \Omega_2$, so the connectedness of Ω gives $\Omega = \Omega_1$ or $\Omega = \Omega_2$. In the first case, (g_n) converges locally uniformly in Ω, and in the second case, (g_n) tends to ∞ locally uniformly.

Let $D \subseteq \Omega$ be an open disc, and let us show that (f_n) has a subsequence which converges in $\mathrm{Hol}(D)$ or tends to ∞ locally uniformly in D. Let $D = D(z_0, R)$. We split into cases:

1. $|f_n(z_0)| \le 1$ for infinitely many values of n: By Schottky's theorem, we get a subsequence (f_{n_j}) such that for any compact $K \subseteq D$, $|f_{n_j}(z)| \le C_K$ for $z \in K$, $j = 1, 2, \dots$. By Montel's theorem, we get a locally uniformly convergent subsequence.

2. $1 < |f_n(z_0)|$ for infinitely many values of n: Then apply Schottkey's theorem and then Montel's theorem to $1/f_n(z) \in \mathrm{Hol}(D)$. We get a subsequence $1/f_{n_k} \to g \in \mathrm{Hol}(D)$ locally uniformly. We have that g is either nonvanishing (then $f_{n_k} \to 1/g$ locally uniformly) or $g \equiv 0$ (then $f_{n_k} \to \infty$ locally uniformly). □

23.3 Corollaries of Picard's great theorem

Recall the Casorati-Weierstrass theorem.

Theorem 23.3 (Casorati-Weierstrass)**.** *Let $a \in \mathbb{C}$, and let $f \in \mathrm{Hol}(\{0 < |z-a| < \delta\})$ have an essential singularity at a. Then the range $f(\{0 < |z-a| < \delta\})$ is dense in $\mathbb{C}$.*

Picard's great theorem is a generalization of this.

Theorem 23.4. *Let $a \in \mathbb{C}$, and let $f \in \mathrm{Hol}(\{0 < |z-a| < \delta\})$ have an essential singularity at a. There exists $w \in \mathbb{C}$ be such that the range $f(\{0 < |z-a| < r\})$ contains $\mathbb{C} \setminus \{w\}$ for all $0 < r \le \delta$.*

Remark 23.1. The function $f(z) = e^{1/z} \neq 0$ has an essential singularity at 0.

We will prove the result next time. Here are some corollaries.

Corollary 23.1. *Let $f \in \mathrm{Hol}(\mathbb{C})$ not be a polynomial. Then for all $R > 0$, f assumes all values in $\mathbb{C}$ with at most 1 exception in $|z| > R$.*

Proof. Apply Picard's great theorem to $f(1/z)$. □

Corollary 23.2. *Let f be meromorphic in $\mathbb{C}$, and suppose f is not a rational function. Then for all $R > 0$, f assumes all values in $\mathbb{C}$ with at most 2 exceptions in $|z| > R$.*

Proof. Assume that f omits 3 distinct values a, b, c in $|z| > R$. Let $g(z) = 1/(f(z) - c)$. Then g removable singularities, so it extends to an entire function. Moreover, g is not a polynomial. g omits the values $1/(a-c)$ and $1/(b-c)$ in $|z| > R$, which contradicts the previous corollary. □

24 Picard's Great Theorem and Fatou's Theorem

24.1 Picard's Great Theorem

Theorem 24.1 (Picard's great theorem)**.** *Let $a \in \mathbb{C}$, and let $f \in \mathrm{Hol}(\{0 < |z-a| < \delta\})$ have an essential singularity at a. There exists $w \in \mathbb{C}$ be such that the range $f(\{0 < |z-a| < r\})$ contains $\mathbb{C} \setminus \{w\}$ for all $0 < r \leq \delta$.*

Proof. We may assume that $a = 0$. Assume that there exists some $\varepsilon > 0$ such that $f \in \mathrm{Hol}(0 < |z| < \varepsilon)$ and $f(0 < |z| < \varepsilon)$ omits 2 distinct values $a, b \in \mathbb{C}$. Let $f_n(z) = f(z/n) \in \mathrm{Hol}(0 < |z| < \varepsilon)$, so $a, b \notin \mathrm{Ran}(f_n)$ for all $n \geq 1$. Apply the Montel-Caratheodory theorem to (f_n) to get a subsequence (f_{n_ν}) such that either (f_{n_ν}) converges locally uniformly in $\mathrm{Hol}(0 < |z| < \varepsilon)$ or $f_n \to \infty$ locally uniformly.

Case 1: Assume that (f_{n_ν}) converges locally uniformly in $\mathrm{Hol}(0 < |z| < \varepsilon)$. Let $K = \{z : |z| = \varepsilon/2\}$. Then $|f_{n_\nu}(z)| \leq C$ for all $z \in K$, $\nu = 1, 2, \ldots$. In other words, $|f(z)| \leq C$ for $|z| = \varepsilon/(2n_\nu) \to 0$. By the maximum principle, f is bounded in a punctured neighborhood of 0, so 0 is a removable singularity for f. This is a contradiction.

Case 2: Assume that $f_{n_\nu} \to \infty$ locally uniformly. Let $g_n(z) = 1/(f_n(z) - a)$. Then g_{n_ν} is a sequence of holomorphic functions with $g_{n_\nu} \to 0$ locally uniformly. Arguing as in Case 1, we get: $g(z) = 1/(f(z) - a)$ has a removable singularity at 0 with $g(0) = 0$. So $f = a + 1/g(z)$ has a pole at 0, which is impossible. □

24.2 Boundary values of harmonic functions in the disc

Theorem 24.2 (Fatou)**.** *Let u be harmonic in D and bounded. Then the radial limits $\lim_{r \to 1^-} u(rz)$ exist for a.e. $z \in \partial D$ (with respect to 1-dimensional) Lebesgue measure on the circle. If $u = f \in \mathrm{Hol}(D)$ and $f(z) = \lim_{r \to 1^-} f(rz)$ vanishes on a set of positive measure (on the circle), then $f \equiv 0$.*

Proof. We may assume that u is real-valued. When $0 \leq r < 1$, let $\mu_r : L^1(\partial D) \to \mathbb{C}$ be the linear, continuous functional given by

$$\mu_r(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(re^{i\varphi}) f(e^{i\varphi})\, d\varphi.$$

We have $|\mu_r(f)| \leq M \|f\|_{L^1}$. Then

$$\|\mu_r\|_{(L^1)^*} = \sup_{0 \neq f \in L^1} \frac{|\mu_r(f)|}{\|f\|_{L^1}} \leq M, \qquad 0 \leq r < 1.$$

We can apply the Banach-Alaoglu theorem[5]: let B be a separable Banach space, and let (Λ_α) be a sequence of linear, continuous functionals $B \to \mathbb{C}$ such that $\|\Lambda_\alpha\|_{B^*} \leq C$ for

[5] The idea of the proof is to let take a countable dense subset (u_ν) of B and use diagonalization to find (Λ_{α_j}) such that $\lim_{j\to\infty} \Lambda_{\alpha_j}(u_\nu)$. Then extend to any $u \in B$ using $\|\Lambda_\alpha\|_{B^*} \leq C$.

all α. Then there exists a subsequence (Λ_{α_j}) such that for all $u \in B$, $(\Lambda_{\alpha_j}(u))$ converges in $\mathbb{C}$. In our case, $B = L^1$, so there exists a sequence $r_k \to 1$ such that for every $f \in L^1$, $\lim_{r_k \to 1} \mu_{r_k}(f)$ exists. Define $\mu(f)$ as this limit. We have $\mu : L^1 \to \mathbb{C}$ is linear, and $\|\mu\|_{(L^1)^*} \leq M$. Thus, $\mu \in (L^1)^*$, the space of linear, continuous functionals on L^1. This space is $L^\infty(D)$; that is, there is a $g \in L^\infty(D)$ such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\varphi}) g(e^{i\varphi})\, d\varphi.$$

We get

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e) u(r_k e^{i\varphi}) f(e^{i\varphi}) \xrightarrow{k \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} g(e^{i\varphi}) f(e^{i\varphi})\, d\varphi.$$

Now $z \mapsto u(r_k z)$ is harmonic in a neighborhood of $|z| \leq 1$, so

$$u(r_k z) = \int P(z, e^{i\varphi}) u(r_k e^{i\varphi})\, d\varphi \qquad \forall k, |z| < 1$$

Let $k \to \infty$. $P(z, e^{i\varphi}) \in L^1(\partial D)$, so

$$u(z) = \int_{-\pi}^{\pi} P(z, e^{i\varphi}) g(e^{i\varphi})\, d\varphi.$$

In other words, u is harmonic and bounded iff u equals the Poisson integral of g for some $g \in L^\infty$. Next, we will show that $\lim_{r \to 1} u(rz) = g(z)$ for a.e. z. $\square$

We will finish the proof next time.

25 Fatou's Theorem and the Riesz-Herglotz Theorem

25.1 Fatou's theorem, continued

Last time, we were in the middle of proving Fatou's theorem.

Theorem 25.1 (Fatou). *Let u be harmonic in D and bounded. Then the radial limits $\lim_{r\to 1^-} u(rz)$ exist for a.e. $z \in \partial D$ (with respect to 1-dimensional) Lebesgue measure on the circle. If $u = f \in \mathrm{Hol}(D)$ and $f(z) = \lim_{r\to 1^-} f(rz)$ vanishes on a set of positive measure (on the circle), then $f \equiv 0$.*

Proof. We have shown that there exists $g \in L^\infty(\partial D)$ such that

$$u = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(z, e^{i\varphi})g(e^{i\varphi})\,d\varphi.$$

Let $e^{i\alpha} \in \partial D$ be a Lebesgue point of g:

$$\frac{1}{2\pi\rho}\int_{\alpha-\rho}^{\alpha+\rho} |g(e^{i\varphi}) - g(e^{i\alpha})|\,d\varphi \to 0.$$

We claim that the radial limit $\lim_{r\to 1} u(re^{i\alpha})$ exists an equals $g(e^{i\alpha})$. This will establish the theorem, as a.e. point in ∂D is a Lebesgue point of g. We can assume that $\alpha = 0$ and that $g(e^{i\alpha}) = 0$ (otherwise consider $u(e^{i\alpha z}) - g(e^{i\alpha})$). Thus,

$$\frac{1}{2\pi\rho}\int_{-\rho}^{\rho} |g(e^{i\varphi})|\,d\varphi \to 0,$$

and we want to show that $u(x) \to 0$ as $x \to 1^-$ along $\mathbb{R}$.

Plugging in the formula for the Poisson kernel,

$$u(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-x^2}{|x-e^{i\varphi}|^2} g(e^{i\varphi})\,d\varphi.$$

The contribution to this integral coming from $\int_{\pi/2\le|\varphi|\le\pi} \to 0$, as $P(x, e^{i\varphi}) \to 0$ uniformly in φ. Estimate the contribution from $|\varphi| \le \pi/2$: Writing $\delta = 1 - x$,

$$P(x, e^{i\varphi}) = \frac{1-(1-\delta)^2}{|x-e^{i\varphi}|^2} = \frac{2\delta - \delta^2}{(x-\cos(\varphi))^2 + \sin^2(\varphi)} \le \frac{2\delta}{\sin^2(\varphi)} \le \frac{2\delta}{\varphi^2}.$$

We get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-x^2}{|x-e^{i\varphi}|^2}|g(e^{i\varphi})|\,d\varphi \le \int_{A\delta\le|\varphi|\le\pi/2} + \int_{|\varphi|\le A\delta}$$

$$\begin{aligned}
&\le \int_{A\delta\le|\varphi|\le\pi/2} \frac{2\delta}{\varphi^2}|g(e^{i\varphi})|\,d\varphi + \int_{|\varphi|\le A\delta} \frac{2\delta-\delta^2}{|x-e^{i\varphi}|^2}|g(e^{i\varphi})|\,d\varphi \\
&\le \int_{A\delta\le|\varphi|\le\pi/2} \frac{2\delta}{\varphi^2}|g(e^{i\varphi})|\,d\varphi + \int_{|\varphi|\le A\delta} \frac{2\delta}{\delta^2}|g(e^{i\varphi})|\,d\varphi \\
&\le \frac{C\delta}{A\delta} + \frac{2}{\delta}\int_{|\varphi|\le A\delta} |g(e^{i\varphi})|\,d\varphi.
\end{aligned}$$

Given $\varepsilon > 0$, take A large so that $C/A \le \varepsilon$ for all $0 < \delta \le \delta_0(\varepsilon)$. For δ small enough, $\int_{|\varphi|\le\pi/2} P(x, e^{i\varphi})|g(e^{i\varphi})|\,d\varphi \le 7\varepsilon$. Thus, $u(x) \to 0$ as $x \to 1^-$. Thus, for a.e. $z \in \partial D$, $\lim_{r\to 1} u(rz)$ exists and equals $g(z)$.

For the latter part of the theorem, assume now that $f \in \mathrm{Hol}(D)$ is bounded. Then for a.e. $z \in \partial D$, $\lim_{r\to 1} f(rz) =: f(z) \in L^\infty(\partial D)$. We claim that if $f(z) = 0$ on a set of positive measure in ∂D, then $f(z) \equiv 0$ in $|z| < 1$. The function $\log|f|$ is subharmonic in D, so

$$r \mapsto \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\varphi})|\,d\varphi$$

is an increasing function. For any $0 < r < 1$, using Fatou's lemma,

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\varphi})|\,d\varphi &\le \limsup_{r\to 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\varphi})|\,d\varphi \\
&\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(e^{i\varphi})|\,d\varphi.
\end{aligned}$$

If $f \not\equiv 0$, we can conclude that the integral $> -\infty$. So $\log|f| \in L^1(\partial D)$, so $\{f = 0\}$ is a Lebesgue null set in ∂D. $\square$

25.2 Representing harmonic functions by measures

We have been looking at functions u such that

$$u(z) = \frac{1}{2\pi}\int_{|w|=1} P(z, w)g(q)\,|dw|$$

for some $g \in L^\infty$. Let's try to replace $g \in L^\infty$ by $g \in L^1$ or by a (Borel, regular, Radon) measure $d\mu$ on ∂D.

Theorem 25.2 (F. Riesz-Herglotz)**.** *Let μ be a measure on ∂D, and let*

$$u = \int_{|w|=1} P(z, w)\,d\mu(w), \qquad |z| < 1.$$

Then u is harmonic in D, and the function $r \mapsto \int_{|z|=1} |u(rz)|\,|dz|$ is bounded on $[0,1)$. If $u_r(z) = u(rz)$, then $u_r \xrightarrow{r\to 1} \mu$ in the following weak sense: for any $\varphi \in C(\partial D)$,

$$\frac{1}{2\pi}\int_{|z|=1} u_r(z)\varphi(z)\,|dz| \xrightarrow{r\ to 1} \int_{|z|=1} \varphi(z)\,d\mu(z).$$

Conversely, let u be harmonic in D such that $\int_{|z|=1} |u(rz)|\,|dz| \leq C$ for all $0 \leq r \leq 1$. Then there exists a unique measure μ on ∂D such that

$$u(z) = \int_{|w|=1} P(z,w)\,d\mu(w) \qquad |z| < 1.$$

Moreover, $u_r \to \mu$ in the same weak sense.

Example 25.1. Let $u \geq 0$ be harmonic. Then the theorem applies, so

$$u(z) = \int_{|w|=1} P(z,w)\,d\mu(w),$$

where μ is a positive measure.

26 Harmonic measures

26.1 The Riesz-Herglotz theorem

Theorem 26.1 (F. Riesz-Herglotz). *u is harmonic in D and*

$$\sup_{0\le r<1}\int_{|z|=1}|u(rz)|\,|dz|\le C<\infty$$

if and only if there exists a measure μ on ∂D such that

$$u(z)=\int_{|w|=1}P(z,w)\,d\mu(w).$$

Proof. Let $u(z)=\int_{|w|=1}P(z,w)\,d\mu(w)$ for $|z|<1$. Then u is harmonic in D, and

$$\begin{aligned}u(re^{it})&=\int_{[-\pi,\pi)}P(re^{it},e^{i\varphi})d\mu(\varphi)\\&=\int_{[-\pi,\pi)}\frac{1-r^2}{1+t^2-2r\cos(t-\varphi)}d\mu(\varphi)\\&=\int_{[-\pi,\pi)}P(re^{i\varphi},e^{it})d\mu(\varphi).\end{aligned}$$

So

$$\begin{aligned}\int_{-\pi}^{\pi}|u(re^{it})|\,dt&\le\int_{-\pi}^{\pi}\left(\int_{[-\pi,\pi)}P(re^{i\varphi},e^{it})\,|d\mu(\varphi)|\,dt\right)\\&=\int_{[-\pi,\pi)}\underbrace{\left(\int_{-\pi}^{\pi}P(re^{i\varphi},e^{it})\,dt\right)}_{=2\pi}|d\mu(\varphi)|\\&\le 2\pi\int_{[-\pi,\pi)}|\,d\mu(\varphi)|.\end{aligned}$$

Check also that if $u_r(z)=u(rz)$, then for all $\psi\in C(\partial D)$,

$$\frac{1}{2\pi}\int_{|z|=1}u_r(z)\psi(z)\,|dz|\to\int_{|z|=1}\psi(z)\,d\mu(z).$$

The left hand side is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_{[-\pi,\pi)}P(re^{it},e^{i\varphi})\,d\mu(\varphi)\right)\psi(e^{it})\,dt=\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}P(re^{it},e^{i\varphi})\psi(e^{it})\,dt\right)\,d\mu(\varphi),$$

where the part in the parentheses on the right is the harmonic extension of $\psi \in C(\overline{D})$, so it converges to $\psi(e^{i\varphi})$ uniformly in φ as $r \to 1$. So this goes to $\int_{[-\pi,\pi)} \psi(e^{i\varphi})\, d\mu(\varphi)$.

Conversely, let u be harmonic in D such that

$$\|u_r\|\|_{L^1(\partial D)} = \int_{-\pi}^{\pi} |u(rz)|\, |dz| \leq C, \qquad 0 \leq r < 1.$$

Here $L^1(\partial D) \subseteq \mathcal{M}(\partial D)$, the space of bounded finite Borel measures on ∂D. The space $\mathcal{M}(\partial D)$ is the dual of $C(\partial D)$. By Banach-Alaoglu, there exists a sequence $r_j \to 1$ and a measure $\mu \in \mathcal{M}(\partial D)$ such that $u_{r_j} \to \mu$ weakly: for any $\psi \in C(\partial D)$,

$$\frac{1}{2\pi} \int_{|z|=1} u_{r_j}(z)\psi(z)|dz| \to \int \psi\, d\mu.$$

Finally, for all j, $u_{r_j}(z)$ is harmonic near $\overline{D}$, so

$$u(r_j z) = \frac{1}{2\pi} \int_{|w|=1} P(z,w)u(r_j w)\, dz.$$

Letting $j \to \infty$, we get

$$u(z) = \int P(z,w)\, d\mu(w).$$

□

Remark 26.1. The measure μ is unique. Let $h^1 = \{u \in H(D) : \int |u(rz)|\, |dz| \leq C\ \forall r\}$. The theorem says that the **Poisson operator** $\mathcal{P} : \mathcal{M}(\partial D) \to h^1$ is a homeomorphism.

Corollary 26.1. *Let $f \in \mathrm{Hol}(D)$ with $\mathrm{Re}(f) \geq 0$. Then there exists a measure $\mu \geq 0$ on ∂D and a constant $c \in \mathbb{R}$ such that*

$$f(z) = ic + \int_{|w|=1} \frac{w+z}{w-z}\, d\mu(w).$$

Proof. By the Riesz-Herglotz theorem applied to $\mathrm{Re}(f) \geq 0$, we write

$$\mathrm{Re}(f(z)) = \int_{|w|=1} \mathrm{Re}\left(\frac{w+z}{w-z}\right)\, d\mu(w).$$

So if

$$g(z) = \int_{|w|=1} \frac{w+z}{w-z}\, d\mu(w),$$

then $g \in \mathrm{Hol}(D)$, and $\mathrm{Re}(f) = \mathrm{Re}(f)$. The result follows. □

26.2 Boundary behavior of harmonic measures

We would like to understand the boundary behavior of $u \in h^1$.

Theorem 26.2. *Let $u \in h^1$, and consider the Lebesgue decomposition of the representing measure μ: $d\mu = f/(2\pi)\,|dz| + d\lambda$, where $f \in L^1(\partial D)$, and $d\lambda$ is singular with respect to $|dz|$.*

1. *Then for a.e. $z \in \partial D$, the radial limit $\lim_{r\to 1} u(rz)$ exists and equals $f(z)$.*
2. *If $d\mu = f/(2\pi)|dz|$ with $f \in L^1$, then $u_r \to f$ in $L^1(\partial D)$.*

We will prove this next time. Here is an application:

Example 26.1 (Problem 12, Analysis qual, Spring 2016)**.** Let u be real, harmonic in D, $u \leq M$, and assume that $\lim_{r\to 1} u(rz) \leq 0$ for a.e. $z \in \partial D$. Show that $u \leq 0$.

Consider $v = M - u \geq 0$, which is harmonic. There exists a measure $\mu \geq 0$ such $v(z) = \int_{|w|=1} P(z,w)\,d\mu(w)$. Writing $d\mu = f/(2\pi)|dz| + d\lambda$, where $f \geq 0$ and $\lambda \geq 0$. By the theorem, $f(z) = \lim_{r\to 1} v(rz) = \lim_{r\to 1}(M - u(rz)) \geq M$. We get

$$v(z) = \underbrace{\int P(z,w)\frac{f}{2\pi}|dw|}_{\geq M} + \underbrace{\int P(z,w)\,d\lambda(w)}_{\geq 0}.$$

So $v \geq M$ in D, and we get $u \leq 0$ in D.

27 Radial Limits of Harmonic Functions on the Disc

27.1 Radial limits of harmonic functions on the disc

Let $\mathcal{P} : \mathcal{M}(\partial D) \to h^1$, the set of all harmonic functions u in D such that $\int_{|z|=1} |u(rz)|\,|dz| \leq C$ for all r, send $\mu \mapsto \mathcal{P}\mu = u$. We showed last time that this is a homeomorphism.

Theorem 27.1. *Let $u \in h^1$, and consider the Lebesgue decomposition of the representing measure μ: $d\mu = f/(2\pi)\,|dz| + d\lambda$, where $f \in L^1(\partial D)$, and $d\lambda$ is singular with respect to $|dz|$.*

1. *Then for a.e. $z \in \partial D$, the radial limit* $\lim_{r\to 1} u(rz)$ *exists and equals $f(z)$.*
2. *If $d\mu = f/(2\pi)|dz|$, is absolutely continuous and $u(z) = \int_{|w|=1} P(z,w)\,d\mu(w)$, then $u_r \to f$ in $L^1(\partial D)$.*

Proof. Write

$$u(z) = \int_{|w|=1} P(z,w)\,d\mu(w) = \int_{[-\pi,\pi)} P(z, r^{i\varphi})\,d\mu(\varphi).$$

Recall that for a.e. $\varphi \in \mathbb{R}$, we have by the Lebesgue differentiation theorem that

$$\frac{1}{\rho}\int_{\varphi-\rho}^{\varphi+\rho} |f(e^{it}) - f(e^{i\varphi})|\,dt \xrightarrow{\rho\to 0} 0,$$

$$\frac{1}{\rho}\int_{[\varphi-\rho,\varphi+\rho]} |d\lambda(t)| \to 0.$$

We claim that if $\varphi \in \mathbb{R}$ is as above, then $\lim_{r\to 1} u(re^{i\varphi})$ exists and equals $f(e^{i\varphi})$. We may assume that $\varphi = 0$ and $f(1) = 0$. Then

$$\frac{1}{\rho}\int_{-\rho}^{\rho} |f(e^{it})|\,dt \to 0, \qquad \frac{1}{\rho}\int_{[-\rho,\rho]} |d\lambda(t)| \to 0.$$

It suffices to show that if $|nu$ is a measure such that $(1/\rho)\int_{[-\rho,\rho]} |d\nu(t)| \to 0$ as $\rho \to 0$, then

$$\int P(x, e^{it})\,d\nu(t) \xrightarrow{x\to 1^-} 0, \qquad x \in \mathbb{R}.$$

Here,

$$\int_{\pi/2\leq|t|\leq\pi} P(x, e^{it})\,d\nu(t)$$

since $P(x, e^{it}) \to 0$ uniformly. Write $\delta = 1 - x$, and consider

$$\int_{|t|\leq\ pi/2} P(x, e^{it})\,d\nu(t) = \int_{\sqrt{c\delta}\leq|t|\leq\pi/2} P(x, e^{it})\,d\nu(t) + \int_{|t|\leq\sqrt{c\delta}} P(x, e^{it})\,d\nu(t).$$

Here, $C > 0$ is a large constant to be chosen later. When $\sqrt{C\delta} \leq |t| \leq |\pi/2|$,

$$P(x, e^{it}) = \frac{1 - x^2}{|x - e^{it}|^2} = \frac{2\delta - \delta^2}{(x - \cos(t))2 + \sin^2(t)} \leq \frac{2\delta}{\sin^2(t)} \leq \frac{\pi^2\delta}{t^2} \leq \frac{\pi^2\delta}{C\delta} = \frac{\pi^2}{C}.$$

Given $\varepsilon > 0$, we get (taking C large but fixed)

$$\left| \int_{\sqrt{C\delta} \leq |t| \leq \pi/2} P(x, e^{it})\, d\nu(t) \right| \leq \varepsilon$$

for all small $\delta > 0$.

Let $\delta_1 = \sqrt{C\delta}$, and let

$$\varphi(t) = P(x, e^{it}) = \frac{1 - x^2}{1 + x^2 - 2x\cos(t)}.$$

Then $\varphi > 0$, φ is even, and φ is decreasing on $[0, \pi]$. It remains to understand

$$\int_{|t| \leq \sqrt{C\delta}} P(x, e^{it})\, d\nu(t) = \int_{|t| \leq \delta_1} \varphi(t)\, d\nu(t).$$

We have

$$\int_{[-\rho,\rho]} |d\nu(t)| \leq \varepsilon\rho, \qquad 0 < \rho \leq \delta_1.$$

Write

$$\varphi(t) = \varphi(\delta_1) + \int_{\delta_1}^{t} \varphi'(s)\, ds = \varphi(\delta_1) - \int_0^{\delta_1} H(s - t)\varphi'(s)\, ds,$$

where

$$H(\tau) = \begin{cases} 1 & \tau > 0 \\ 0 & \tau \leq 0 \end{cases}$$

is the Heaviside function. Consider

$$\int_{[0,\delta_1]} \varphi(t)\, d\nu(t) = \varphi(\delta_1) \underbrace{\int_{[0,\delta-1]} d\nu(t)}_{\leq \varepsilon\delta_1} - \int_{[0,\delta_1]} \left(\int_0^{\delta_1} H(s - t)\varphi'(s)\, ds \right) d\nu(t).$$

Then

$$\begin{aligned} \left| \int_{[0,\delta_1]} \varphi(t)\, d\nu(t) \right| &\leq \varphi(\delta_1)\varepsilon\delta_1 - \int_0^{\delta_1} \varphi'(s) \left(\int_{[0,\delta_1]} H(s - t)\, |d\nu(t)| \right) ds \\ &\leq \varphi(\delta_1)\varepsilon\delta_1 - \int_0^{\delta_1} \varphi'(s) \underbrace{\left(\int_{[0,s]} |d\nu(t)| \right)}_{\leq \varepsilon s} ds \end{aligned}$$

Integrate by parts.

$$\begin{aligned}
&\le \varphi(\delta_1)\varepsilon\delta - 1 - \varepsilon\,[\varphi(s)s]_0^{\delta_1} + \varepsilon \int_0^{\delta_1} \varphi(s)\,ds \\
&= \varepsilon \int_0^{\delta_1} \varphi(s)\,ds \\
&\le \varepsilon.
\end{aligned}$$

The contribution of $[-\delta, 0]$ is estimated similarly. We get

$$u(x) = \int P(x, e^{it})\,d\nu(t) \xrightarrow{x \to 1^-} 0.$$

For the 2nd part of the theorem, given $\varepsilon >$, let $\psi \in C(\partial D)$ be such that $\|f - \psi\|_{L^1} \le \varepsilon$. If we write $u = \mathcal{P}f$, then

$$\|(\mathcal{P}f)_r - f\|_{L^1} \le \underbrace{\|(\mathcal{P}f)_r - (\mathcal{P}\psi)_r\|_{L^1}}_{\le \|\mathcal{P}(f-\psi)\|_{h^1} \le \|f-\psi\|_{L^1} \le \varepsilon} + \underbrace{\|(\mathcal{P}\psi)_r - \psi\|_{L^1}}_{\to 0 \text{ uniformly on } \partial D} + \varepsilon.$$

We get $u_r = (\mathcal{P}f)_r \to f$ in L^1. □

27.2 The Riesz-Riesz theorem

Let $H^1 = \mathrm{Hol}(D) \cap h^1$ (the **Hardy space**). It can be show that the representing measure of and H^1 function is absolutely continuous.

Theorem 27.2 (F. and M. Riesz[6]). *Let μ be a measure on ∂D such that $\int_{[0,2\pi)} e^{int}\,d\mu(t) = 0$ for $n = 1, 2, \dots$ (i.e. the negative Fourier coefficients of μ vansish). Then μ is absolutely continuous.*

Proof. Here is the idea. Let $f = \mathcal{P}\mu \in h^1$. The vanishing of the Fourier coefficients implies that $f \in \mathrm{Hol}(D)$. So μ is absolutely continuous. □

[6]These two were brothers. This is the only collaboration between them.